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16. Summary/Notes <i>This fifth chapter, on the Fundamentals of Plasma Physics, introduces the concepts of phase space and distribution function, necessary for a statistical description of plasma phenomena. A derivation is presented of the Boltzmann equation, but without explicitly deriving an expression for the collision term, and of the Vlasov equation. The relaxation model for the collision term, also known as Krook's model, is introduced.</i>			
17. Remarks			

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CHAPTER 5

ELEMENTS OF PLASMA KINETIC THEORY

1. INTRODUCTION

A plasma is a system containing a very large number of interacting charged particles. For the analysis of a system consisting of a very large number of particles it is appropriate to use a *statistical approach*. In this chapter we present the basic elements of kinetic theory, introducing the concepts of *phase space* and *distribution function* necessary for a statistical description. All physically interesting information about the system is contained in the distribution function. From a knowledge of the distribution function, the average values of the various physical quantities of interest, which can be considered as functions of the particle velocities, can be deduced systematically and used to describe the macroscopic behavior of the plasma.

The differential kinetic equation satisfied by the distribution function, known as the *Boltzmann equation*, is deduced in section 5. At this stage, the effects of collisions are incorporated into this kinetic equation only through a general, unspecified collision term. In Chapter 21 we shall deduce explicit expressions for the collision term, in particular for the Boltzmann collision integral and for the Fokker-Planck collision term. Only a simple approximate expression for the collision term is presented at this point, the so-called *relaxation model* or *Krook collision term*. The *Vlasov equation* for a plasma is introduced in the last section.

2. PHASE SPACE

At any instant of time each particle in the plasma can be localized by a position vector $\underline{r}$ drawn from the origin of a coordinate system to the center of mass of the particle. In the Cartesian frame of reference we have

$$\underline{r} = \hat{x}\underline{x} + \hat{y}\underline{y} + \hat{z}\underline{z} \quad (2.1)$$

where $\hat{x}$, $\hat{y}$ and $\hat{z}$ denote unit vectors along the axes x , y and z , respectively. The linear velocity of the center of mass of the particle can be represented by

$$\underline{v} = \frac{d\underline{r}}{dt} = \hat{x}v_x + \hat{y}v_y + \hat{z}v_z \quad (2.2)$$

whith $v_x = dx/dt$, $v_y = dy/dt$ and $v_z = dz/dt$.

In analogy with *configuration space* defined by the position coordinates (x, y, z) , it is convenient to introduce the *velocity space* defined by the coordinates (v_x, v_y, v_z) . In this space the velocity vector $\underline{v}$ can be considered as a position vector drawn from the origin of the coordinate system (v_x, v_y, v_z) , as indicated in Fig. 1.

2.1 - Single-particle phase space

From the point of view of classical mechanics the instantaneous dynamic state of each particle can be specified by its position and velocity. It is convenient, therefore, to consider the *phase space* defined by the six coordinates x, y, z, v_x, v_y, v_z .

Fig. 1 - Position vectors (a) in configuration space and (b) in velocity space .

In this six-dimensional space the dynamical state of each particle is appropriately represented by a single point. The coordinates $(\underline{r}, \underline{v})$ of the representative point give the position and velocity of the particle. When the particle moves, its representative point describes a trajectory in phase space. At each instant of time the dynamical state of a system of N particles is represented by N points in phase space.

2.2 - Many-particle phase space

The phase space just defined, often called μ -space, is the phase space for a single particle, in contrast with the many particle phase space or Γ -space for the whole system of particles. In the latter, a system consisting of N particles, with no internal degrees of freedom, is represented by a single point in a $6N$ -dimensional space defined by the $3N$ position coordinates $(\underline{r}_1, \underline{r}_2, \dots, \underline{r}_N)$ and the $3N$ velocity coordinates $(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_N)$. Thus, a point in Γ -space corresponds to a single microscopic state for the whole system of particles. This many-particle phase space is often used in statistical mechanics. The single-particle phase space is the one normally used in plasma physics and is the space to be considered in what follows.

2.3 - Volume elements

A small element of volume in configuration space is represented by $d^3r = dx \, dy \, dz$. This differential element of volume should not be taken literally as a mathematically infinitesimal quantity but as a finite element of volume, sufficiently large to contain a very large number of particles, yet sufficiently small in comparison with the characteristic lengths associated with the spatial variation of physical parameters of interest such as, for example, pressure and temperature. In a gas containing 10^{20} molecules/ m^3 , for example, if we take $d^3r \sim 10^{-12} \text{ m}^3$, which in a macroscopic scale can be considered as a point, there are still 10^8 molecules

inside d^3r . Plasmas that do not allow a choice of differential elements of volume as indicated, cannot be analysed statistically.

When we refer to a particle as being situated inside d^3r , at $\underline{r}$, it is meant that the x-coordinate of the particle lies between x and $x + dx$, the y-coordinate between y and $y + dy$, and the z-coordinate between z and $z + dz$, that is, inside the volume element $d^3r = dx dy dz$ situated around the terminal point of the position vector $\underline{r} = \hat{x}x + \hat{y}y + \hat{z}z$. It is important to note that the particles localized inside d^3r , at $\underline{r}$, may have completely arbitrary velocities which would be represented by scattered points in velocity space.

A small element of volume in velocity space is represented by $d^3v = dv_x dv_y dv_z$. For a particle to be included in d^3v , around the terminal point of the velocity vector $\underline{v}$, its v_x - component of the velocity must lie between v_x and $v_x + dv_x$, the v_y - component between v_y and $v_y + dv_y$, and the v_z - component between v_z and $v_z + dv_z$. The differential elements of volume d^3r and d^3v are schematically represented in Fig. 2.

In phase space (μ -space) a differential element of volume may be imagined as a six-dimensional cube, represented by

$$d^3r d^3v = dx dy dz dv_x dv_y dv_z \quad (2.3)$$

as shown schematically in Fig. 3. Note that inside $d^3r d^3v$, at the position $(\underline{r}, \underline{v})$ in phase space, there are only the particles inside d^3r around $\underline{r}$ whose velocities lie inside d^3v about $\underline{v}$. The number of representative points inside the volume element $d^3r d^3v$ is, in general, a function of time and of the position of this element in phase space. It is important to note that the coordinates $\underline{r}$ and $\underline{v}$ of phase space are considered to be independent variables, since they represent the position of individual volume elements (containing many particles) in phase space.

Fig. 2 - (a) The element of volume $d^3r = dx dy dz$ around the terminal point of $\underline{r}$, in configuration space, and (b) the element of volume $d^3v = dv_x dv_y dv_z$, in velocity space, around the terminal point of $\underline{v}$.

Fig. 3 - Schematic representation of the element of volume $d^3r d^3v$ in the six-dimensional phase space, around the point $(\underline{r}, \underline{v})$.

3. DISTRIBUTION FUNCTION

Let $d^6n_\alpha(\underline{r}, \underline{v}, t)$ denote the number of particles of type α inside the element of volume $d^3r d^3v$ around the coordinates $(\underline{r}, \underline{v})$ of phase space, at the instant of time t . The *distribution function* in phase space, $f_\alpha(\underline{r}, \underline{v}, t)$, is defined as the density of representative points of the type α particles in phase space, that is,

$$f_\alpha(\underline{r}, \underline{v}, t) = \frac{d^6n_\alpha(\underline{r}, \underline{v}, t)}{d^3r d^3v} \quad (3.1)$$

It is assumed that the density of representative points in phase space does not vary rapidly from one element of volume to the neighboring element, so that $f_{\alpha}(\underline{r}, \underline{v}, t)$ can be considered as a continuous function of its arguments. According to its definition $f_{\alpha}(\underline{r}, \underline{v}, t)$ is also a positive and finite function at any instant of time. In an element of volume $d^3r d^3v$, whose velocity coordinates (v_x, v_y, v_z) are very large, the number of representative points is relatively small since, in any macroscopic system, there must be relatively few particles with very large velocities. Physical considerations require, therefore, that $f_{\alpha}(\underline{r}, \underline{v}, t)$ tends to zero as the velocity becomes infinitely large.

The distribution function is, in general, a function of the position vector $\underline{r}$. When this is the case the corresponding plasma is said to be *inhomogeneous*. In the absence of external forces, however, a plasma initially inhomogeneous reaches, in the course of time, an equilibrium state as a result of the mutual interactions amongst the various particles of the plasma. In this *homogeneous* state the distribution function does not depend on the position vector $\underline{r}$.

In velocity space the distribution function can be *anisotropic*, when it depends on the orientation of the velocity vector $\underline{v}$, or *isotropic*, when it does not depend on the orientation of the velocity vector $\underline{v}$ but only on its magnitude, i.e., on the speed $v = |\underline{v}|$.

The description of different types of plasmas requires the use of inhomogeneous or homogeneous, as well as anisotropic or isotropic distribution functions. A plasma in *thermal equilibrium*, for example, is characterized by a homogeneous, isotropic and time-independent distribution function.

In a statistical sense the distribution function provides a complete description of the system under consideration. Knowing $f_{\alpha}(\underline{r}, \underline{v}, t)$ we can deduce all the macroscopic variables of physical interest for the particles of type α . One of the primary problems of kinetic theory consists in determining the form of the distribution function for a given system. The differential equation that governs the temporal and spatial variation of the distribution function under given conditions, known as the Boltzmann equation, will be considered in section 5.

4. NUMBER DENSITY AND AVERAGE VELOCITY

The *number density*, $n_{\alpha}(\underline{r}, t)$, is defined as the number of particles of type α per unit volume, irrespective of velocity. It can be obtained by integrating $d^6n_{\alpha}(\underline{r}, \underline{v}, t)$ over all velocity space and dividing the result by the element of volume d^3r of configuration space,

$$n_{\alpha}(\underline{r}, t) = \frac{1}{d^3r} \int_{\underline{v}} d^6n_{\alpha}(\underline{r}, \underline{v}, t) \quad (4.1)$$

or, using the definition (3.1),

$$n_{\alpha}(\underline{r}, t) = \int_{\underline{v}} f_{\alpha}(\underline{r}, \underline{v}, t) d^3v \quad (4.2)$$

The single integral sign indicated here represents in fact a *triple* integral extending over all velocity space, that is, over each one of the variables v_x, v_y, v_z from $-\infty$ to $+\infty$. For convenience and simplification of notation only a single integral sign will be used, being implicit the fact that the integral extends over all of velocity space.

The *average velocity* $\underline{u}_{\alpha}(\underline{r}, t)$ is defined as the macroscopic flow velocity of the particles of type α in the neighborhood of the position vector $\underline{r}$ at the instant t . In order to relate $\underline{u}_{\alpha}(\underline{r}, t)$ with the distribution function $f_{\alpha}(\underline{r}, \underline{v}, t)$, consider the particles of type α contained in the volume element $d^3r d^3v$ about $(\underline{r}, \underline{v})$ at the instant t , which we have denoted by $d^6n_{\alpha}(\underline{r}, \underline{v}, t)$. The average velocity of the particles of type α can be obtained as follows. First we multiply $d^6n_{\alpha}(\underline{r}, \underline{v}, t)$ by the particle velocity $\underline{v}$, next we integrate over all possible velocities, and finally we divide the result by the total number of type α particles contained in d^3r irrespective of velocity. Therefore,

$$\underline{u}_{\alpha}(\underline{r}, t) = \frac{1}{n_{\alpha}(\underline{r}, t) d^3r} \int_{\underline{v}} \underline{v} d^6n_{\alpha}(\underline{r}, \underline{v}, t) \quad (4.3)$$

The procedure just described is the usual statistical definition of average values. Using the definition of $f_{\alpha}(\underline{r}, \underline{v}, t)$ given in (3.1), we obtain

$$\underline{u}_{\alpha}(\underline{r}, t) = \frac{1}{n_{\alpha}(\underline{r}, t)} \int_{\underline{v}} \underline{v} f_{\alpha}(\underline{r}, \underline{v}, t) d^3v \quad (4.4)$$

Note that both $n_{\alpha}(\underline{r}, t)$ and $\underline{u}_{\alpha}(\underline{r}, t)$ are *macroscopic* variables which depend only upon the coordinates $\underline{r}$ and t .

A systematic method for deducing the macroscopic variables (such as momentum flux, pressure, temperature, heat flux and so on) in terms of the distribution function, is presented in a formal manner in Chapter 6.

5. THE BOLTZMANN EQUATION

In order to calculate the average values of the particle physical properties and the macroscopic variables of interest it is necessary to know the distribution function $f_{\alpha}(\underline{r}, \underline{v}, t)$ for the system under consideration. The dependence of the distribution function on the independent variables $\underline{r}$, $\underline{v}$ and t is governed by an equation known as the *Boltzmann equation*. We present in this section

a derivation of the collisionless Boltzmann equation and the general form it takes when the effects of the particle interactions are taken into account, without explicitly deriving any particular expression for the collision term.

5.1 - Collisionless Boltzmann equation

Recall that

$$d^6n_{\alpha}(\underline{r}, \underline{v}, t) = f_{\alpha}(\underline{r}, \underline{v}, t) d^3r d^3v \quad (5.1)$$

represents the number of particles of type α which, at the instant of time t , are situated within the volume element $d^3r d^3v$ of phase space, about the coordinates $(\underline{r}, \underline{v})$. Suppose that each particle is subjected to an *external force* $\underline{F}$. In the absence of the mutual particle interactions, a particle of type α with coordinates $(\underline{r}, \underline{v})$ in phase space, at the instant of time t , will be found after a time interval dt , in the new coordinates $(\underline{r}', \underline{v}')$, such that

$$\underline{r}'(t + dt) = \underline{r}(t) + \underline{v} dt \quad (5.2)$$

$$\underline{v}'(t + dt) = \underline{v}(t) + \underline{a} dt \quad (5.3)$$

where $\underline{a} = \underline{F}/m_{\alpha}$ is the particle acceleration, and m_{α} is the mass of a particle of type α .

Thus, all particles of type α inside the volume element $d^3r \ d^3v$ of phase space, about $(\underline{r}, \underline{v})$ at the instant of time t , will occupy a new volume element $d^3r' \ d^3v'$, about $(\underline{r}', \underline{v}')$ after the time interval dt (see Fig. 4). Since we are considering the same particles at t and at $t + dt$, we must have in the absence of collisions,

$$f_{\alpha}(\underline{r}', \underline{v}', t + dt) \ d^3r' \ d^3v' = f_{\alpha}(\underline{r}, \underline{v}, t) \ d^3r \ d^3v \quad (5.4)$$

Fig. 4 - In the absence of collisions the particles within the volume element $d^3r \ d^3v$ about $(\underline{r}, \underline{v})$, at an instant t , will occupy after a time interval dt a new volume element $d^3r' \ d^3v'$, about $(\underline{r}', \underline{v}')$.

The element of volume $d^3r d^3v$ may become distorted in shape as the result of the motion of the particles. The relation between the new element of volume, $d^3r' d^3v'$, and the initial one, $d^3r d^3v$, is given by

$$d^3r' d^3v' = |J| d^3r d^3v \quad (5.5)$$

where J stands for the Jacobian of the transformation from the initial coordinates $(\underline{r}, \underline{v})$ to the final coordinates $(\underline{r}', \underline{v}')$. It will be shown in the next subsection that for the transformation defined by Eqs. (5.2) and (5.3) we have $|J| = 1$, so that

$$d^3r' d^3v' = d^3r d^3v \quad (5.6)$$

and Eq. (5.4) becomes

$$\left[f_{\alpha}(\underline{r}', \underline{v}', t + dt) - f_{\alpha}(\underline{r}, \underline{v}, t) \right] d^3r d^3v = 0 \quad (5.7)$$

The first term on the left of Eq. (5.7) can be expanded in a Taylor series about $f_{\alpha}(\underline{r}, \underline{v}, t)$ as follows

$$\begin{aligned} f_{\alpha}(\underline{r} + \underline{v} dt, \underline{v} + \underline{a} dt, t + dt) = f_{\alpha}(\underline{r}, \underline{v}, t) + \left[\frac{\partial f_{\alpha}}{\partial t} + \right. \\ \left. + \left(v_x \frac{\partial f_{\alpha}}{\partial x} + v_y \frac{\partial f_{\alpha}}{\partial y} + v_z \frac{\partial f_{\alpha}}{\partial z} \right) \right] dt \end{aligned}$$

$$+ \left[a_x \frac{\partial f_\alpha}{\partial v_x} + a_y \frac{\partial f_\alpha}{\partial v_y} + a_z \frac{\partial f_\alpha}{\partial v_z} \right] dt \quad (5.8)$$

neglecting terms of order $(dt)^2$ and higher. Using the del operator notation

$$\underline{\nabla} = \underline{\hat{x}} \frac{\partial}{\partial x} + \underline{\hat{y}} \frac{\partial}{\partial y} + \underline{\hat{z}} \frac{\partial}{\partial z} \quad (5.9)$$

and, in a similar way, defining a del operator in velocity space by

$$\underline{\nabla}_v = \underline{\hat{x}} \frac{\partial}{\partial v_x} + \underline{\hat{y}} \frac{\partial}{\partial v_y} + \underline{\hat{z}} \frac{\partial}{\partial v_z} \quad (5.10)$$

we obtain from Eq. (5.8)

$$f_\alpha(\underline{r} + \underline{v} dt, \underline{v} + \underline{a} dt, t + dt) = f_\alpha(\underline{r}, \underline{v}, t) + \left[\frac{\partial f_\alpha}{\partial t} + \underline{v} \cdot \underline{\nabla} f_\alpha + \underline{a} \cdot \underline{\nabla}_v f_\alpha \right] dt \quad (5.11)$$

Substituting this result into Eq. (5.7) gives

$$\frac{\partial f_\alpha}{\partial t} + \underline{v} \cdot \underline{\nabla} f_\alpha + \underline{a} \cdot \underline{\nabla}_v f_\alpha = 0 \quad (5.12)$$

which is the Boltzmann equation in the absence of collisions.

This equation can be rewritten as

$$\frac{Df_{\alpha}}{Dt} = 0 \quad (5.13)$$

where the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla} + \underline{a} \cdot \underline{\nabla}_v \quad (5.14)$$

represents the *total* derivative with respect to time, in phase space. Eq. (5.13) is a statement of the conservation of the density of representative points in phase space. If we move along with a representative point in phase space and observe the density of representative points f_{α} in its neighborhood, we find that this density remains constant in time. This result is known as *Liouville's theorem*. Note that it applies only to the special case in which collisions, as well as radiation losses and processes of production and loss of particles are unimportant.

5.2 - Jacobian of the transformation in phase space

To determine the Jacobian of the transformation defined in Eqs. (5.2) and (5.3), recall that, from its definition, we have

$$J = \frac{\partial(\underline{r}', \underline{v}')}{\partial(\underline{r}, \underline{v})} = \frac{\partial(x', y', z', v'_x, v'_y, v'_z)}{\partial(x, y, z, v_x, v_y, v_z)} \quad (5.15)$$

which corresponds to the determinant of the 6 x 6 matrix

$$(J) = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \dots & \frac{\partial v'_z}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \dots & \frac{\partial v'_z}{\partial y} \\ \dots & \dots & \dots & \dots \\ \frac{\partial x'}{\partial v_z} & \frac{\partial y'}{\partial v_z} & \dots & \frac{\partial v'_z}{\partial v_z} \end{pmatrix} \quad (5.16)$$

We can separate the external force $\underline{F}$ into two parts,

$$\underline{F} = \underline{F}' + q_\alpha (\underline{v} \times \underline{B}) \quad (5.17)$$

where $\underline{F}'$ is a velocity-independent force and the second term is the velocity-dependent force due to an externally applied magnetic field $\underline{B}$, the only velocity-dependent force that may concern us in this treatment. The partial derivatives appearing in (J) are

$$\begin{aligned} \frac{\partial x'_i}{\partial x_j} &= \delta_{ij}, & \frac{\partial v'_i}{\partial x_j} &= \frac{1}{m_\alpha} \frac{\partial F'_i}{\partial x_j} dt \\ \frac{\partial x'_i}{\partial v_j} &= \delta_{ij} dt, & \frac{\partial v'_i}{\partial v_j} &= \delta_{ij} + \frac{q_\alpha}{m_\alpha} \frac{\partial}{\partial v_j} (\underline{v} \times \underline{B})_i dt \end{aligned} \quad (5.18)$$

where Eqs. (5.2), (5.3) and (5.17) have been used, and where

$x_{i,j} = x, y, z$ and $v_{i,j} = v_x, v_y, v_z$. The symbol δ_{ij} is the Kronecker delta. The matrix (5.16) can be written in the form

$$(J) = \begin{pmatrix} (J)_1 & (J)_2 \\ (J)_3 & (J)_4 \end{pmatrix} \quad (5.19)$$

where the $(J)_i$'s, with $i = 1, 2, 3, 4$, represent the following 3×3 submatrices

$$(J)_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.20)$$

$$(J)_2 = \begin{pmatrix} \frac{1}{m_\alpha} \frac{\partial F'_x}{\partial x} dt & \frac{1}{m_\alpha} \frac{\partial F'_y}{\partial x} dt & \frac{1}{m_\alpha} \frac{\partial F'_z}{\partial x} dt \\ \frac{1}{m_\alpha} \frac{\partial F'_x}{\partial y} dt & \frac{1}{m_\alpha} \frac{\partial F'_y}{\partial y} dt & \frac{1}{m_\alpha} \frac{\partial F'_z}{\partial y} dt \\ \frac{1}{m_\alpha} \frac{\partial F'_x}{\partial z} dt & \frac{1}{m_\alpha} \frac{\partial F'_y}{\partial z} dt & \frac{1}{m_\alpha} \frac{\partial F'_z}{\partial z} dt \end{pmatrix} \quad (5.21)$$

$$(J)_3 = \begin{pmatrix} dt & 0 & 0 \\ 0 & dt & 0 \\ 0 & 0 & dt \end{pmatrix} \quad (5.22)$$

$$(J)_4 = \begin{pmatrix} 1 & -\frac{q_\alpha}{m_\alpha} B_z dt & \frac{q_\alpha}{m_\alpha} B_y dt \\ \frac{q_\alpha}{m_\alpha} B_z dt & 1 & -\frac{q_\alpha}{m_\alpha} B_x dt \\ -\frac{q_\alpha}{m_\alpha} B_y dt & \frac{q_\alpha}{m_\alpha} B_x dt & 1 \end{pmatrix} \quad (5.23)$$

Neglecting terms of order $(dt)^2$ it can be verified that $|J| = 1$. Thus, up to and including the terms of first order in the infinitesimal dt , we have,

$$d^3r' d^3v' = d^3r d^3v \quad (5.24)$$

which is the result (5.6) used in the previous subsection.

5.3 - Effects of particle interactions

When the effects due to the interactions between the various particles constituting the plasma are taken into account, Eq. (5.12) needs to be modified. As a result of collisions during the time interval dt , some of the particles of type α which were initially within the volume element $d^3r d^3v$ of phase space may be removed from it, and particles of type α initially outside this volume element may end up inside it. This is indicated schematically in Fig. 5.

Fig. 5 - Schematic representation of the motion of the volume element $d^3r d^3v$ in phase space, showing particles entering and leaving this volume element, as a result of collisions during the time interval dt .

Generally, therefore, the number of particles of type α inside $d^3r d^3v$ about the coordinates $(\underline{r}, \underline{v})$ at an instant t , will be different from the number of particles of type α inside this same volume element about the coordinates $(\underline{r}', \underline{v}')$ at the instant $t + dt$. We shall denote this net gain or loss of particles of type α , as a result of collisional interactions during the time interval dt , in the volume element $d^3r d^3v$ of phase space, by

$$\left(\frac{\delta f_{\alpha}}{\delta t}\right)_{\text{coll}} d^3r d^3v dt \quad (5.25)$$

where $(\delta f_{\alpha} / \delta t)_{\text{coll}}$ represents the rate of change of $f_{\alpha}(\underline{r}, \underline{v}, t)$ due to collisions. Thus, when collisions are considered, Eq. (5.7) becomes

$$\left[f_{\alpha}(\underline{r}', \underline{v}', t + dt) - f_{\alpha}(\underline{r}, \underline{v}, t) \right] d^3r d^3v = \left(\frac{\delta f_{\alpha}}{\delta t}\right)_{\text{coll}} d^3r d^3v dt \quad (5.26)$$

and the following modified form of Eq. (5.12) results

$$\frac{\partial f_{\alpha}}{\partial t} + \underline{v} \cdot \underline{\nabla} f_{\alpha} + \underline{a} \cdot \underline{\nabla}_{\underline{v}} f_{\alpha} = \left(\frac{\delta f_{\alpha}}{\delta t}\right)_{\text{coll}} \quad (5.27)$$

Using the total time derivative operator, defined in Eq. (5.14), we can rewrite this equation in compact form as

$$\frac{Df_{\alpha}}{Dt} = \left(\frac{\delta f_{\alpha}}{\delta t} \right)_{\text{coll}} \quad (5.28)$$

This equation is obviously incomplete, since the precise form of the collision term is not known. In the following section we will consider a very simple expression for the collision term, known as the *Krook* or *relaxation model*. More elaborated expressions, such as the *Boltzmann collision integral* and the *Fokker-Planck collision term* will be considered in chapter 21.

6. RELAXATION MODEL FOR THE COLLISION TERM

A very simple method for taking into account collision effects between the particles is provided by the relaxation model. In this model it is assumed that the effect of collisions is to restore a situation of *local equilibrium*, characterized by the local equilibrium distribution function $f_{0\alpha}(\underline{r}, \underline{v})$. In the absence of external forces, it assumes that a situation initially not in equilibrium, described by a distribution function $f_{\alpha}(\underline{r}, \underline{v}, t)$ different from $f_{0\alpha}(\underline{r}, \underline{v})$, reaches a local equilibrium condition exponentially with time, as result of collisions, with a *relaxation time* τ . This relaxation time is of the order of the time between collisions and may also be written as ν^{-1} where ν represents a *relaxation collision frequency*. This model was developed by *Krook* and can be written as

$$\left(\frac{\delta f_{\alpha}}{\delta t}\right)_{\text{coll}} = - \frac{(f_{\alpha} - f_{0\alpha})}{\tau} \quad (6.1)$$

According to this expression for the collision term, when $f_{\alpha} = f_{0\alpha}$ we have $(\delta f_{\alpha} / \delta t)_{\text{coll}} = 0$, so that in a state of local equilibrium the distribution function is not altered as a result of collisions between the particles.

In order to put in evidence the physical meaning of the relaxation model, let us consider the Boltzmann equation with this collision term, in the absence of external forces and spatial gradients, and when $f_{0\alpha}$ and τ are time-independent,

$$\frac{\partial f_{\alpha}}{\partial t} = - \frac{(f_{\alpha} - f_{0\alpha})}{\tau} \quad (6.2)$$

which may also be written as

$$\frac{\partial f_{\alpha}}{\partial t} + \frac{f_{\alpha}}{\tau} = \frac{f_{0\alpha}}{\tau} \quad (6.3)$$

This simple inhomogeneous differential equation has $C e^{-t/\tau}$ as the homogeneous solution, where C is a constant, and $f_{0\alpha}$ as a particular solution. Therefore, the complete solution is

$$f_{\alpha}(\underline{v}, t) = f_{0\alpha} + \left[f_{\alpha}(\underline{v}, 0) - f_{0\alpha} \right] \exp (-t/\tau) \quad (6.4)$$

Thus, the difference between f_α and $f_{0\alpha}$ decreases exponentially with time at a rate governed by the relaxation collision frequency $\nu = 1/\tau$.

The relaxation model for the collision term has proved to be useful and, in some cases, leads to results almost identical to the ones obtained using the Boltzmann collision integral. It is particularly applicable for a weakly ionized plasma in which only the collisions between charged and neutral particles are important. However, it oversimplifies the entire relaxation phenomena and does not predict correctly the different relaxation collision frequencies for the various physical quantities of interest, such as average velocity, momentum, and energy. According to the relaxation model, the macroscopic physical variables approach equilibrium at the same rate ν . A detailed analysis of the collision process, however, shows that this is not the case and that the relaxation time for various macroscopic variables varies to some extent. For nonrelativistic velocities, while the relaxation time for the average velocity and the momentum are found to be the same, approximately τ , that of the average thermal energy is approximately $(m_b/2m_\alpha)\tau$. Hence, for collisions between electrons and neutral particles, the relaxation time for the kinetic energy of the electrons is *longer* than that for the average velocity by a factor which is of the order of the ratio of the mass of the neutral particle to the electron mass. The relaxation model is, therefore, strictly applicable only to the cases of collisions between particles of the same mass. In spite of this limitation, the Krook model is still useful partly because of its simplicity and partly because it gives a first approximation to the problem under consideration.

7. THE VLASOV EQUATION

A very useful approximate way to describe the dynamics of a plasma is to consider that the motions of the plasma particles are governed by the applied external fields *plus* the macroscopic average internal fields, smoothed in space and time, due to the presence and motion of all plasma particles. The problem of obtaining the macroscopically smoothed internal electromagnetic fields, however, is still a complex one and requires that a *self-consistent* solution be obtained.

The Vlasov equation is a partial differential equation that describes the evolution of the distribution function in time, and which directly incorporates the macroscopically smoothed internal electromagnetic fields. It may be obtained from the Boltzmann equation with the collision term $(\delta f_{\alpha} / \delta t)_{\text{coll}}$ equal to zero, but including the *internal smoothed fields* in the force term,

$$\frac{\partial f_{\alpha}}{\partial t} + \underline{v} \cdot \underline{\nabla} f_{\alpha} + \frac{1}{m_{\alpha}} \left[\underline{F}_{\text{ext}} + q_{\alpha} (\underline{E}_i + \underline{v} \times \underline{B}_i) \right] \cdot \underline{\nabla}_v f_{\alpha} = 0 \quad (7.1)$$

Here $\underline{F}_{\text{ext}}$ represents the external forces, including the Lorentz force associated with any *externally* applied electric and magnetic fields, and $\underline{E}_i$ and $\underline{B}_i$ are *internal* smoothed electric and magnetic fields due to the presence and motion of all charged particles inside the plasma. In order that the internal electromagnetic fields $\underline{E}_i$ and $\underline{B}_i$

be consistent with the charge and current densities existing in the plasma itself, they must satisfy Maxwell equations

$$\underline{\nabla} \cdot \underline{E}_i = \rho_c / \epsilon_0 \quad (7.2)$$

$$\underline{\nabla} \cdot \underline{B}_i = 0 \quad (7.3)$$

$$\underline{\nabla} \times \underline{E}_i = - \partial \underline{B}_i / \partial t \quad (7.4)$$

$$\underline{\nabla} \times \underline{B}_i = \mu_0 (\underline{J} + \epsilon_0 \partial \underline{E}_i / \partial t) \quad (7.5)$$

with the plasma charge density ρ_c and the plasma current density $\underline{J}$ given by the formulas

$$\rho_c(\underline{r}, t) = \sum_{\alpha} n_{\alpha}(\underline{r}, t) q_{\alpha} = \sum_{\alpha} q_{\alpha} \int_{\underline{v}} f_{\alpha}(\underline{r}, \underline{v}, t) d^3v \quad (7.6)$$

$$\begin{aligned} \underline{J}(\underline{r}, t) &= \sum_{\alpha} n_{\alpha}(\underline{r}, t) q_{\alpha} \underline{u}_{\alpha}(\underline{r}, t) = \\ &= \sum_{\alpha} q_{\alpha} \int_{\underline{v}} \underline{v} f_{\alpha}(\underline{r}, \underline{v}, t) d^3v \end{aligned} \quad (7.7)$$

the summations being over the different charged particle species in the plasma. Here $\underline{u}_{\alpha}(\underline{r}, t)$ denotes the macroscopic *average velocity* for the particles of type α , given in (4.4).

Eqs. (7.1) to (7.7) constitute a complete set of *self-consistent* equations to be solved simultaneously. Assuming values for $\underline{E}_i(\underline{r}, t)$ and $\underline{B}_i(\underline{r}, t)$, Eq. (7.1) can be solved to yield $f_\alpha(\underline{r}, \underline{v}, t)$ for the various different species; using the calculated $f_\alpha(\underline{r}, \underline{v}, t)$'s in (7.6) and (7.7), leads to values for the charge and current densities in the plasma, which can be substituted into Maxwell equations and solved for $\underline{E}_i(\underline{r}, t)$ and $\underline{B}_i(\underline{r}, t)$; these values are then plugged back into the Vlasov equation, and so on. Hence, a self-consistent solution can be obtained for the single particle distribution function $f_\alpha(\underline{r}, \underline{v}, t)$.

Although the Vlasov equation does not explicitly include a collision term in its right-hand side and, hence, does not take into account short-range correlations, it is not so restrictive as it may appear, since a significant part of the effects of the particle interactions has already been included in the Lorentz force term.

PROBLEMS

5.1 - Consider a system of particles uniformly distributed in space, with a constant particle number density n_0 , and characterized by a velocity distribution function $f(v)$ such that

$$f(v) = K_0 \neq 0 \quad \text{for} \quad |v_i| \leq v_0 \quad (i = x, y, z)$$

$$f(v) = 0 \quad \text{otherwise,}$$

where K_0 is a positive constant. Determine the value of K_0 in terms of n_0 and v_0 .

5.2 - Consider the following *two-dimensional* Maxwellian distribution function

$$f(v_x, v_y) = n_0 \left(\frac{m}{2\pi kT} \right) \exp \left[- \frac{m(v_x^2 + v_y^2)}{2kT} \right]$$

(a) Verify that n_0 represents correctly the particle number density, that is, the number of particles per unit area.

(b) Draw, in three dimensions, the surface for this distribution function, plotting f in terms of v_x and v_y . Sketch, on this surface, curves of constant v_x , curves of constant v_y , and curves of constant f .

- 5.3 - The electrons, inside a system of two coaxial magnetic mirrors, can be described by the so-called *loss-cone distribution function*

$$f(\underline{v}) = \frac{n_0}{(\pi^{3/2} \alpha_{\perp}^2 \alpha_{\parallel})} \left(\frac{v_{\perp}}{\alpha_{\perp}} \right)^2 \exp \left[- \left(\frac{v_{\perp}}{\alpha_{\perp}} \right)^2 - \left(\frac{v_{\parallel}}{\alpha_{\parallel}} \right)^2 \right]$$

where $v_{\parallel}$ and $v_{\perp}$ denote the electron velocities in the directions parallel and perpendicular to the magnetic bottle axis, respectively, and where $\alpha_{\perp}^2 = 2k T_{\perp}/m$ and $\alpha_{\parallel}^2 = 2k T_{\parallel}/m$.

(a) Verify that the number density of the electrons in the magnetic bottle is given by n_0 .

(b) Justify the applicability of the loss-cone distribution function to a magnetic mirror bottle, by analysing its dependence on $v_{\perp}$ and $v_{\parallel}$. Sketch, in three dimensions, the surface for $f(\underline{v})$ as a function of $v_{\perp}$ and $v_{\parallel}$.

- 5.4 - Consider the motion of charged particles, in one dimension only, in an electric potential $V(x)$. Show, by direct substitution, that a function of the form

$$f = f\left(\frac{1}{2} m v^2 + qV\right)$$

is a solution of the Boltzmann equation under steady state conditions.

5.5 - (a) Show that the Boltzmann equation, in cylindrical coordinates, becomes

$$\begin{aligned} \frac{\partial f}{\partial t} + \dot{r} \frac{\partial f}{\partial r} + \dot{\phi} \frac{\partial f}{\partial \phi} - r \dot{\phi}^2 \frac{\partial f}{\partial \dot{r}} + \frac{2\dot{r}\dot{\phi}}{r} \frac{\partial f}{\partial \dot{\phi}} + \dot{z} \frac{\partial f}{\partial z} + \\ + \frac{1}{m} \left(F_r \frac{\partial f}{\partial \dot{r}} + \frac{F_\phi}{r} \frac{\partial f}{\partial \dot{\phi}} + F_z \frac{\partial f}{\partial \dot{z}} \right) = \left(\frac{\delta f}{\delta t} \right)_{\text{coll}} \end{aligned}$$

where $F_r = m\ddot{r}$, $F_\phi = mr\ddot{\phi}$ and $F_z = m\ddot{z}$.

(b) Show, by direct substitution, that, in the presence of an azimuthally symmetric magnetic field (in the z -direction), a function of the form

$$f = f\left(\frac{1}{2}mv^2, mr^2\dot{\phi} + qrA_\phi\right)$$

is a solution of the Boltzmann equation under steady conditions, where $p_\phi = mr^2\dot{\phi} + qrA_\phi$ is the constant canonical momentum, and where A_ϕ denotes the ϕ -component of the magnetic potential, defined such that $\underline{B} = \underline{\nabla} \times \underline{A}$.

5.6 - Show that the Vlasov equation for a homogeneous plasma under the influence of a uniform external magnetostatic field, $\underline{B}_0$, in the equilibrium state, is satisfied by any homogeneous distribution function, $f(v_\parallel, v_\perp)$ which is cylindrically symmetric with respect to the magnetostatic field.

5.7 - The entropy of a system can be expressed, in terms of the distribution function, as

$$S = -k \int_r \int_v f \ln(f) d^3v d^3r$$

Show that, for a system which obeys the collisionless Boltzmann equation, the total time derivative of the entropy vanishes.

5.8 - Consider a one-dimensional harmonic oscillator whose energy is given by

$$E = \frac{p^2}{2m} + \frac{1}{2} kx^2$$

where p denotes its linear momentum and x its displacement coordinate. Show that the trajectory described by the representative point of the oscillator, in phase space, is an ellipse.