1. Classification <i>INPE-COM.4/RPE</i> <i>C.D.U.: 533.9</i>		2. Period	4. Distribution Criterion internal ☐ external ☒
3. Key Words (selected by the author) <i>MOMENTS OF THE BOLTZMANN EQUATION</i> <i>CONSERVATION EQUATIONS</i> <i>COLD PLASMA MODEL</i> <i>WARM PLASMA MODEL</i>			
5. Report No. <i>INPE-1471-RPE/022</i>	6. Date <i>April, 1979</i>	7. Revised by <i>J. Sobral</i> <i>J.H.A. Sobral</i>	
8. Title and Sub-title <i>MACROSCOPIC TRANSPORT EQUATIONS</i>		9. Authorized by <i>Parada</i> <i>Nelson de Jesus Parada</i> <i>Director</i>	
10. Sector <i>DCE/GIO</i>	Code	11. No. of Copies <i>13</i>	
12. Authorship <i>J.A. Bittencourt</i>		14. No. of Pages <i>51</i>	
13. Signature of first author <i>Bittencourt</i>		15. Price	
16. Summary/Notes <i>This chapter presents a derivation of the macroscopic transport equations, necessary for the description of plasmas as fluids, in a way which emphasizes their origin in the kinetic theory. The equations of conservation of mass, of momentum, and of energy, are obtained from a general transport equation, and are identified with the hierarchy of moments of the Boltzmann equation. The approximations inherent in the cold and warm plasma models are explicitly indicated.</i>			
17. Remarks			

CHAPTER 8

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CHAPTER 8

MACROSCOPIC TRANSPORT EQUATIONS

1. MOMENTS OF THE BOLTZMANN EQUATION

In the previous chapters we have seen that the macroscopic variables of physical interest for a plasma, such as number density n_α , mean velocity $\underline{u}_\alpha$, temperature T_α , and so on, can be calculated if we know the distribution function for the system under consideration. For the case of a system in thermal equilibrium, we have calculated, in Chapter 7, several of these macroscopic parameters using the Maxwell-Boltzmann distribution function. In principle the distribution function for a system not in equilibrium can be obtained by solving the Boltzmann equation. However, the solution of the Boltzmann equation is generally a matter of great difficulty.

We will see, in this chapter, that it is not necessary to solve the Boltzmann equation for the distribution function, in order to determine the macroscopic variables of physical interest. The differential equations governing the temporal and spatial variation of these macroscopic variables can be derived directly from the Boltzmann equation without solving it. These differential equations are known as the *macroscopic transport equations*, and their solutions, under certain assumptions, gives us directly the macroscopic variables.

The macroscopic variables are all *moments of the distribution function* and the transport equations satisfied by these variables can be obtained by taking the various *moments of the Boltzmann equation*. The first three moments of the Boltzmann equation obtained by multiplying it by m_α , $m_\alpha v$ and $m_\alpha v^2/2$, respectively, and integrating over all of velocity space, give us the equation of conservation of mass, the equation of conservation of momentum and the equation of conservation of energy. However, at each stage of the hierarchy of moments of the Boltzmann equation, the resulting set of transport equations is not complete in the sense that the number of equations is not sufficient to determine all the macroscopic variables that appear in them. Each time a higher moment of the Boltzmann equation is calculated in an attempt to obtain a complete set of transport equations, a new macroscopic variable appears. It is necessary, therefore, to truncate the system of transport equations at some stage of the hierarchy and to introduce a simplifying assumption concerning the highest moment of the distribution function that appears in the system. Thus, with such simplifying approximation, we obtain a complete set of transport equations sufficient to determine all the macroscopic variables appearing in the system. Since a plasma is composed of more than one species of particles (electrons, ions, neutral particles) there is, consequently, a system of transport equations for each species.

There are several different complete sets of transport equations (or hydrodynamic equations) that can be formed, depending on the assumptions considered. Amongst the possible complete systems of macroscopic equations, there are two which are widely used and which characterize the so-called *cold* and *warm plasma models*. The equations that describe these two simple models, and the corresponding approximations, will be discussed in sections 6 and 7 of this chapter.

2. GENERAL TRANSPORT EQUATION

We derive now a general partial differential equation that describes the temporal and spatial variation of the physically relevant macroscopic parameters. Let $x(\underline{v})$ represent some physical property of the particles in the plasma, which may be, in general, a function of the velocity of the particles. Since the average value of $x(\underline{v})$ is obtained by multiplying the distribution function by the property $x(\underline{v})$, integrating the product over all of velocity space and dividing the result by the particle number density, the differential equation governing the temporal and spatial variation of the average value of $x(\underline{v})$ can be obtained in a similar way by multiplying the Boltzmann equation by the function $x(\underline{v})$ and integrating the resulting equation over all of velocity space.

Consider, therefore, the Boltzmann equation for the particles of type α , in the general form

$$\frac{\partial f_{\alpha}}{\partial t} + \underline{v} \cdot \underline{\nabla} f_{\alpha} + \underline{a} \cdot \underline{\nabla}_v f_{\alpha} = \left(\frac{\delta f_{\alpha}}{\delta t} \right)_{\text{coll}} \quad (2.1)$$

As indicated, we now multiply each term by $\chi(\underline{v})$ and integrate the resulting equation over all of velocity space to obtain

$$\begin{aligned} \int_v \chi \frac{\partial f_{\alpha}}{\partial t} d^3v + \int_v \chi \underline{v} \cdot \underline{\nabla} f_{\alpha} d^3v + \int_v \chi \underline{a} \cdot \underline{\nabla}_v f_{\alpha} d^3v &= \\ &= \int_v \chi \left(\frac{\delta f_{\alpha}}{\delta t} \right)_{\text{coll}} d^3v \end{aligned} \quad (2.2)$$

We proceed next to evaluate separately each of the terms in Eq. (2.2).

The *first term* of (2.2) may be rewritten as

$$\int_{\mathbf{v}} \chi \frac{\partial f_{\alpha}}{\partial t} d^3v = \frac{\partial}{\partial t} \int_{\mathbf{v}} \chi f_{\alpha} d^3v - \int_{\mathbf{v}} f_{\alpha} \frac{\partial \chi}{\partial t} d^3v \quad (2.3)$$

since the limits of integration do not depend upon the space and time variables, and, therefore, the partial time derivative can be taken inside or outside the integral sign. The last integral in (2.3) vanishes since $\chi(\underline{v})$ does not depend upon t . Using the definition of average values, as given in chapter 6, we obtain

$$\int_{\mathbf{v}} \chi \frac{\partial f_{\alpha}}{\partial t} d^3v = \frac{\partial}{\partial t} (n_{\alpha} \langle \chi \rangle_{\alpha}) \quad (2.4)$$

Similarly, for the *second term* of (2.2) we can write

$$\begin{aligned} \int_{\mathbf{v}} \chi \underline{v} \cdot \underline{\nabla} f_{\alpha} d^3v &= \underline{\nabla} \cdot \int_{\mathbf{v}} \underline{v} \chi f_{\alpha} d^3v - \int_{\mathbf{v}} f_{\alpha} \underline{v} \cdot \underline{\nabla} \chi d^3v - \\ &- \int_{\mathbf{v}} f_{\alpha} \chi \underline{\nabla} \cdot \underline{v} d^3v \end{aligned} \quad (2.5)$$

The term involving $\underline{\nabla} \cdot \underline{v}$ is zero since $\underline{r}$, $\underline{v}$ and t are independent variables, as well as the term involving $\underline{\nabla} \chi$, since $\chi(\underline{v})$ does not depend upon the space variables. Thus, the second term of (2.2) becomes

$$\int_{\underline{v}} \chi \underline{v} \cdot \underline{\nabla} f_{\alpha} d^3v = \underline{\nabla} \cdot (n_{\alpha} \langle \chi \underline{v} \rangle_{\alpha}) \quad (2.6)$$

For the *third term* of (2.2) we have, in a similar way,

$$\begin{aligned} \int_{\underline{v}} \chi \underline{a} \cdot \underline{\nabla}_{\underline{v}} f_{\alpha} d^3v &= \int_{\underline{v}} \underline{\nabla}_{\underline{v}} \cdot (\underline{a} \chi f_{\alpha}) d^3v - \int_{\underline{v}} f_{\alpha} \underline{a} \cdot \underline{\nabla}_{\underline{v}} \chi d^3v - \\ &- \int_{\underline{v}} f_{\alpha} \chi \underline{\nabla}_{\underline{v}} \cdot \underline{a} d^3v \end{aligned} \quad (2.7)$$

The last integral in (2.7) vanishes if we assume that

$$\underline{\nabla}_{\underline{v}} \cdot \underline{a} = \frac{1}{m_{\alpha}} \underline{\nabla}_{\underline{v}} \cdot \underline{F} = 0 \quad (2.8)$$

that is, if the force component F_i is independent of the corresponding velocity component v_i , for $i = x, y, z$. Note that this restriction does not exclude the force due to a magnetic field, $\underline{F} = q_{\alpha} \underline{v} \times \underline{B}$, since in this case F_i is still independent of v_i . For the x -component, for example, we have

$$F_x = q_\alpha (v_y B_z - v_z B_y) \quad (2.9)$$

which is independent of v_x , and the same holds true for the other two components.

The first integral in the right-hand side of (2.7) consists of a sum of three triple integrals,

$$\begin{aligned} \int_V \nabla_v \cdot (\underline{a} \times \underline{f}_\alpha) d^3v &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\partial}{\partial v_x} (a_x \times f_\alpha) dv_x dv_y dv_z + \\ &+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\partial}{\partial v_y} (a_y \times f_\alpha) dv_x dv_y dv_z + \\ &+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\partial}{\partial v_z} (a_z \times f_\alpha) dv_x dv_y dv_z \quad (2.10) \end{aligned}$$

For each one of these triple integrals we have the result

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\partial}{\partial v_x} (a_x \times f_\alpha) dv_x dv_y dv_z = \int_{-\infty}^{+\infty} dv_y dv_z (a_x \times f_\alpha) \Big|_{-\infty}^{+\infty} = 0 \quad (2.11)$$

since $f_\alpha(\underline{r}, \underline{v}, t)$ must be zero when v becomes infinitely large, as there are no particles with infinite velocity. Consequently, the first integral in the right-hand side of (2.7) vanishes. Therefore,

$$\int_V \chi \underline{a} \cdot \underline{\nabla}_v f_\alpha d^3v = - n_\alpha \langle \underline{a} \cdot \underline{\nabla}_v \chi \rangle_\alpha \quad (2.12)$$

Combining the results contained in Eqs. (2.4), (2.6) and (2.12) we obtain the *general transport equation*,

$$\begin{aligned} \frac{\partial}{\partial t} (n_\alpha \langle \chi \rangle_\alpha) + \underline{\nabla} \cdot (n_\alpha \langle \chi \underline{v} \rangle_\alpha) - \\ - n_\alpha \langle \underline{a} \cdot \underline{\nabla}_v \chi \rangle_\alpha = \left[\frac{\delta}{\delta t} (n_\alpha \langle \chi \rangle_\alpha) \right]_{\text{coll}} \end{aligned} \quad (2.13)$$

where the term in the right-hand side denotes the time rate of change of the quantity χ per unit volume, for the particles of type α , due to collisional interactions with all other particles in the plasma,

$$\left[\frac{\delta}{\delta t} (n_\alpha \langle \chi \rangle_\alpha) \right]_{\text{coll}} \equiv \int_V \chi \left(\frac{\delta f_\alpha}{\delta t} \right)_{\text{coll}} d^3v \quad (2.14)$$

The equations to be derived in the subsequent sections of this chapter are very general and are not specifically dependent on any particular form of the collision term $(\delta f_\alpha / \delta t)_{coll}$. A derivation of the general transport equation for the case when the property x is a function of $\underline{r}$, $\underline{v}$ and t is included in Problem 8.6.

3. CONSERVATION OF MASS

3.1 - Derivation of the continuity equation

The transport equation (2.13) is a general expression and applies to any arbitrary function $x(\underline{v})$. The equation of continuity, or of conservation of mass, can be obtained by taking $x = m_\alpha$ in (2.13). Hence, for this case, we have

$$\langle x \rangle_\alpha = m_\alpha \quad (3.1a)$$

$$\langle \chi \tilde{v} \rangle_{\alpha} = m_{\alpha} \langle \tilde{v} \rangle_{\alpha} = m_{\alpha} u_{\alpha} \quad (3.1b)$$

$$\tilde{\nabla}_{\tilde{v}} \chi = \tilde{\nabla}_{\tilde{v}} m_{\alpha} = 0 \quad (3.1c)$$

The substitution of these results into the general transport equation gives the *continuity equation*,

$$\frac{\partial \rho_{\alpha}}{\partial t} + \tilde{\nabla} \cdot (\rho_{\alpha} \tilde{u}_{\alpha}) = S_{\alpha} \quad (3.2)$$

where $\rho_{\alpha} = n_{\alpha} m_{\alpha}$ represents the mass density, and where the collision term S_{α} , defined by,

$$S_{\alpha} \equiv m_{\alpha} \int_V \left(\frac{\delta f_{\alpha}}{\delta t} \right)_{\text{coll}} d^3v = \left(\frac{\delta \rho_{\alpha}}{\delta t} \right)_{\text{coll}} \quad (3.3)$$

represents the rate per unit volume at which particles of type α (with mass m_{α}) are produced or lost as a result of collisional interactions. Contributions to this term are due to processes of creation or destruction of particles such as ionization, recombination, attachment, charge transfer, and so on. In the absence of interactions leading to the creation and destruction of particles of type α , the collision term (3.3) is equal to zero, since in the collision process the mass is conserved. For the case when $S_{\alpha} = 0$, the continuity equation reduces to

$$\frac{\partial \rho_{\alpha}}{\partial t} + \nabla \cdot (\rho_{\alpha} \underline{u}_{\alpha}) = 0 \quad (3.4)$$

Dividing each term in (3.4) by m_{α} , the continuity equation can be written in terms of the number density n_{α} , as

$$\frac{\partial n_{\alpha}}{\partial t} + \nabla \cdot (n_{\alpha} \underline{u}_{\alpha}) = 0 \quad (3.5)$$

The equation of conservation of electric charge follows from (3.5) by multiplying it by the charge q_{α} ,

$$\frac{\partial \rho_{c\alpha}}{\partial t} + \nabla \cdot \underline{j}_{\alpha} = 0 \quad (3.6)$$

where $\rho_{c\alpha} = n_{\alpha} q_{\alpha}$ is the charge density and $\underline{j}_{\alpha} = \rho_{c\alpha} \underline{u}_{\alpha}$ is the charge current density.

3.2 - Derivation by the method of fluid dynamics

The continuity equation can also be derived by the method of fluid dynamics since $n_{\alpha}(\underline{r}, t)$ and $\underline{u}_{\alpha}(\underline{r}, t)$ are macroscopic variables. Consider a volume V in the fluid, limited by the closed surface S and let $d\underline{S} = \hat{n} dS$ be an element of area on this surface, such that the unit normal vector $\hat{n}$ points outwards, as shown in Fig. 1.

Fig. 1 - Closed surface S surrounding the arbitrary volume V inside the fluid, and the element of area $d\underline{S} = \hat{n} dS$ pointing outwards.

The average number of particles of type α that leave the volume V through the element of area $d\vec{S}$ per unit time, is given by

$$n_{\alpha} \vec{u}_{\alpha} \cdot d\vec{S} \quad (3.7)$$

Therefore, the number of particles of type α that leave the volume V through the whole closed surface S per unit time, is obtained by integrating expression (3.7) over the whole surface, that is,

$$\oint_S n_{\alpha} \vec{u}_{\alpha} \cdot d\vec{S} \quad (3.8)$$

On the other hand, the total number of particles of type α contained in V , at any instant of time, is given by

$$\int_V n_{\alpha} d^3r \quad (3.9)$$

If we consider that there are no production or loss of particles of type α inside the volume V , then the number of particles of type α leaving the volume V , given in (3.8), is associated with a decrease in time of the number of particles of type α inside V . We must have, therefore,

$$\oint_S n_{\alpha} \vec{u}_{\alpha} \cdot d\vec{S} = - \frac{\partial}{\partial t} \int_V n_{\alpha} d^3r \quad (3.10)$$

Using Gauss' divergence theorem, we can write

$$\oint_S n_\alpha \underline{u}_\alpha \cdot d\underline{S} = \int_V \underline{\nabla} \cdot (n_\alpha \underline{u}_\alpha) d^3r \quad (3.11)$$

and Eq. (3.10) becomes

$$\int_V \left[\frac{\partial n_\alpha}{\partial t} + \underline{\nabla} \cdot (n_\alpha \underline{u}_\alpha) \right] d^3r = 0 \quad (3.12)$$

This result must be valid for any arbitrary volume V , which implies that the integrand of (3.12) vanishes identically. Hence, we obtain the continuity equation in the absence of processes of production and loss of particles of type α , as given in (3.5).

3.3 - The collision term

Let us consider now the form of the collision term S_α for some mechanisms of production and loss of particles in plasmas. The processes leading to production and loss of particles are usually related to inelastic collisions, such as those involving ionization, recombination or electron attachment, for example.

The effect of *ionization* can be included in the continuity equation through a rate coefficient for ionization k_i , defined such that the number of electrons produced per unit time, is given by $k_i n_e$.

An important process leading to the loss of electrons and ions in a plasma is ion-electron *recombination*. Let α denote the recombination coefficient, determined experimentally. The rate of recombination of the electrons is proportional to the product of the electron number density and the number density of the ions present. Assuming there is only one species of ions present, we have $n_i = n_e$, and the loss term for the electrons, due to recombination, can be written as αn_e^2 .

Another important mechanism for the loss of electrons in a plasma is the process of electron *attachment*. In this case, the rate of loss of electrons is proportional to the product of the electron number density and the neutral particle number density. In a weakly ionized plasma the number density of the neutral particles can be considered to be approximately constant, and the loss term for the electrons by attachment can be written as βn_e , where β is the attachment collision frequency, determined experimentally.

For these inelastic collision mechanisms just described, the collision term S_α is given by

$$S_e = m_e (k_i n_e - \alpha n_e^2 - \beta n_e) \quad (3.13)$$

4. CONSERVATION OF MOMENTUM

4.1 - Derivation of the equation of motion

In order to derive the momentum transport equation, we replace $\chi(\underline{v})$ by $m_\alpha \underline{v}$ in the general transport equation (2.13). Taking $\underline{v} = \underline{c}_\alpha + \underline{u}_\alpha$ and noting that $\langle \underline{c}_\alpha \rangle = 0$, the terms of the general transport equation become

$$\frac{\partial}{\partial t} (\rho_\alpha \langle \underline{v} \rangle_\alpha) = \rho_\alpha \frac{\partial \underline{u}_\alpha}{\partial t} + \underline{u}_\alpha \frac{\partial \rho_\alpha}{\partial t} \quad (4.1a)$$

$$\begin{aligned} \underline{\nabla} \cdot (\rho_\alpha \langle \underline{v} \underline{v} \rangle_\alpha) &= \underline{\nabla} \cdot \left[\rho_\alpha (\underline{u}_\alpha \underline{u}_\alpha + \underline{u}_\alpha \langle \underline{c}_\alpha \rangle + \langle \underline{c}_\alpha \rangle \underline{u}_\alpha + \langle \underline{c}_\alpha \underline{c}_\alpha \rangle) \right] \\ &= \underline{\nabla} \cdot (\rho_\alpha \underline{u}_\alpha \underline{u}_\alpha + \rho_\alpha \langle \underline{c}_\alpha \underline{c}_\alpha \rangle) \end{aligned} \quad (4.1b)$$

$$\begin{aligned} -n_\alpha \langle \underline{F} \cdot \underline{\nabla}_v \underline{v} \rangle_\alpha &= -n_\alpha \left(F_x \frac{\partial}{\partial v_x} + F_y \frac{\partial}{\partial v_y} + F_z \frac{\partial}{\partial v_z} \right) \langle \underline{v} \rangle_\alpha \\ &= -n_\alpha \langle F_x \hat{x} + F_y \hat{y} + F_z \hat{z} \rangle_\alpha \\ &= -n_\alpha \langle \underline{F} \rangle \end{aligned} \quad (4.1c)$$

Substituting each of Eqs. (4.1), into (2.13), results in the momentum equation

$$\rho_{\alpha} \frac{\partial \underline{u}_{\alpha}}{\partial t} + \underline{u}_{\alpha} \frac{\partial \rho_{\alpha}}{\partial t} + \nabla \cdot (\rho_{\alpha} \underline{u}_{\alpha} \underline{u}_{\alpha}) + \nabla \cdot (\rho_{\alpha} < \underline{c}_{\alpha} \underline{c}_{\alpha} >) - n_{\alpha} < \underline{F} >_{\alpha} = \underline{A}_{\alpha} \quad (4.2)$$

where $\underline{A}_{\alpha}$ denotes the collision term

$$\underline{A}_{\alpha} \equiv m_{\alpha} \int_V \underline{v} \left(\frac{\delta f_{\alpha}}{\delta t} \right)_{\text{coll}} d^3v = \left[\frac{\delta (\rho_{\alpha} \underline{u}_{\alpha})}{\delta t} \right]_{\text{coll}} \quad (4.3)$$

The expression $\rho_{\alpha} < \underline{c}_{\alpha} \underline{c}_{\alpha} >$ is the kinetic pressure dyad, $\underline{p}_{\alpha}$, defined in Eq. (6.6.2). Therefore,

$$\nabla \cdot (\rho_{\alpha} < \underline{c}_{\alpha} \underline{c}_{\alpha} >) = \nabla \cdot \underline{p}_{\alpha} \quad (4.4)$$

The third term in the left-hand side of (4.2) can be expanded as follows

$$\begin{aligned} \nabla \cdot (\rho_{\alpha} \underline{u}_{\alpha} \underline{u}_{\alpha}) &= \frac{\partial}{\partial x} (\rho_{\alpha} u_{\alpha x} \underline{u}_{\alpha}) + \frac{\partial}{\partial y} (\rho_{\alpha} u_{\alpha y} \underline{u}_{\alpha}) + \frac{\partial}{\partial z} (\rho_{\alpha} u_{\alpha z} \underline{u}_{\alpha}) \\ &= \rho_{\alpha} u_{\alpha x} \frac{\partial \underline{u}_{\alpha}}{\partial x} + \rho_{\alpha} u_{\alpha y} \frac{\partial \underline{u}_{\alpha}}{\partial y} + \rho_{\alpha} u_{\alpha z} \frac{\partial \underline{u}_{\alpha}}{\partial z} + \end{aligned}$$

$$\begin{aligned}
 & + \underline{u}_{\alpha} \frac{\partial (\rho_{\alpha} u_{\alpha x})}{\partial x} + \underline{u}_{\alpha} \frac{\partial (\rho_{\alpha} u_{\alpha y})}{\partial y} + \underline{u}_{\alpha} \frac{\partial (\rho_{\alpha} u_{\alpha z})}{\partial z} \\
 & = \rho_{\alpha} (\underline{u}_{\alpha} \cdot \underline{\nabla}) \underline{u}_{\alpha} + \underline{u}_{\alpha} \left[\underline{\nabla} \cdot (\rho_{\alpha} \underline{u}_{\alpha}) \right] \quad (4.5)
 \end{aligned}$$

Substituting Eqs. (4.4) and (4.5) into (4.2), and using the continuity equation (3.2), we obtain

$$\rho_{\alpha} \left[\frac{\partial \underline{u}_{\alpha}}{\partial t} + (\underline{u}_{\alpha} \cdot \underline{\nabla}) \underline{u}_{\alpha} \right] + \underline{\nabla} \cdot \underline{p}_{\alpha} - n_{\alpha} \langle \underline{F} \rangle_{\alpha} = \underline{A}_{\alpha} - \underline{u}_{\alpha} S_{\alpha} \quad (4.6)$$

For the terms within brackets in this last equation we can use the total (or substantial) time derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}_{\alpha} \cdot \underline{\nabla} \quad (4.7)$$

which corresponds to the time variation observed in a frame of reference moving with the mean velocity $\underline{u}_{\alpha}$. If the electromagnetic Lorentz force, and the gravitational force are considered, the last term in the left-hand side of (4.6) becomes

$$- n_{\alpha} \langle \vec{F} \rangle_{\alpha} = - n_{\alpha} q_{\alpha} (\vec{E} + \vec{u}_{\alpha} \times \vec{B}) - n_{\alpha} m_{\alpha} \vec{g} \quad (4.8)$$

where the fields $\vec{E}$ and $\vec{B}$, in this equation, represent smoothed macroscopic fields. The equation of motion can, therefore, be written as

$$\rho_{\alpha} \frac{D\vec{u}_{\alpha}}{Dt} = n_{\alpha} q_{\alpha} (\vec{E} + \vec{u}_{\alpha} \times \vec{B}) + \rho_{\alpha} \vec{g} - \nabla \cdot \vec{p}_{\alpha} + \vec{A}_{\alpha} - \vec{u}_{\alpha} S_{\alpha} \quad (4.9)$$

Physically, this equation expresses the fact that the time rate of change of the mean momentum, in each fluid element, is due to the external forces applied in the fluid, to the shear (viscosity) and pressure forces of the fluid itself, and to the internal forces associated with the collisional interactions. Thus, the equation of motion establishes the condition necessary to guarantee conservation of momentum, just as the continuity equation establishes the condition necessary to guarantee conservation of mass (or number of particles).

In Chapter 6 we have seen that the term $-\nabla \cdot \vec{p}_{\alpha}$ represents the force exerted in a unit volume of the plasma, due to the random variation in the peculiar velocities of the plasma particles. This force per unit volume includes forces associated with the scalar pressure and tangential shear forces (viscous forces).

In many cases, the effects of viscosity is relatively unimportant in plasmas and the non-diagonal terms of $\underline{\underline{p}}_\alpha$ can, then, be neglected. Furthermore, in the special case when the distribution of peculiar velocities is isotropic, the diagonal terms of $\underline{\underline{p}}_\alpha$ are all equal and correspond to a scalar kinetic pressure p_α . Thus, neglecting viscosity effects and considering an isotropic velocity distribution, we have $\underline{\underline{p}}_\alpha = \frac{1}{3} p_\alpha$, and the force per unit volume becomes $-\nabla \cdot \underline{\underline{p}}_\alpha = -\nabla p_\alpha$, according to (6.6.18).

With these simplifying approximations, and neglecting collisions leading to production or loss of particles ($S_\alpha = 0$), the momentum equation becomes

$$\rho_\alpha \frac{D\mathbf{u}_\alpha}{Dt} = n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \rho_\alpha \mathbf{g} - \nabla p_\alpha + \mathbf{A}_\alpha \quad (4.10)$$

The momentum conservation equation can also be derived using a fluid-dynamics approach, in a way similar to the derivation of the mass conservation equation presented in section 3.2 .

4.2 - The collision term

The symbol $\tilde{A}_\alpha$ denotes the rate of change of mean momentum per unit volume, due to collisional interactions. As a consequence of conservation of the total momentum in an elastic collision, the change in the momentum of one of the particles must be equal and opposite to the change in momentum of the other particle participating in the collision event. This means that, for collisions involving particles of the same species, there is no variation in the total momentum per unit volume, and, therefore, in this case, $\tilde{A}_\alpha = 0$. However, for a fluid composed of particles of different species, as in a plasma, the collision term $\tilde{A}_\alpha$ is not zero in general. For collisions between electrons and neutral particles there is a net momentum transfer from the electron gas to the neutral gas. Collisions between electrons and ions also modify the total momentum of the electron gas. Therefore, for the case of collisions between particles of different species, a collision term must be included in the equation of conservation of momentum.

An expression often used for the term of momentum transfer by collisions is

$$\underline{A}_{\alpha} = - \rho_{\alpha} \sum_{\beta} \nu_{\alpha\beta} (\underline{u}_{\alpha} - \underline{u}_{\beta}) \quad (4.11)$$

which assumes that the force, per unit volume, exerted on the particles of type α due to collisions with particles of some other type β ($\beta \neq \alpha$) is proportional to the difference between the mean velocities $\underline{u}_{\alpha}$ and $\underline{u}_{\beta}$. The proportionality constant $\nu_{\alpha\beta}$ (which has dimensions of sec^{-1}) is called the *collision frequency for momentum transfer* between the particles of type α and those of type β . Since the total momentum must be conserved during a collision, we must have

$$\rho_{\alpha} \nu_{\alpha\beta} (\underline{u}_{\alpha} - \underline{u}_{\beta}) + \rho_{\beta} \nu_{\beta\alpha} (\underline{u}_{\beta} - \underline{u}_{\alpha}) = 0 \quad (4.12)$$

The collision frequencies $\nu_{\alpha\beta}$ and $\nu_{\beta\alpha}$ satisfy, therefore, the following important relationship:

$$\rho_{\alpha} \nu_{\alpha\beta} = \rho_{\beta} \nu_{\beta\alpha} \quad (4.13)$$

The collision term $\underline{A}_{\alpha}$, defined in Eq. (4.3), will be considered in more detail in Chapter 21. We will see, then, that the expression (4.11) is not generally valid, although this result is obtained when the difference between the mean velocities of the various particle species in the plasma is relatively small and when each species of particles has a Maxwellian velocity distribution.

5. CONSERVATION OF ENERGY

5.1 - Derivation of the energy transport equation

To derive the energy transport equation, we substitute $\chi(\underline{v})$ by $m_\alpha v^2/2$ in the general transport equation (2.13). In this case, we have

$$\begin{aligned} n_\alpha \langle \chi \rangle_\alpha &= \frac{1}{2} \rho_\alpha \langle v^2 \rangle_\alpha \\ &= \frac{1}{2} \rho_\alpha \langle c_\alpha^2 \rangle + \frac{1}{2} \rho_\alpha u_\alpha^2 \\ &= \frac{3}{2} p_\alpha + \frac{1}{2} \rho_\alpha u_\alpha^2 \end{aligned} \quad (5.1)$$

$$\begin{aligned} \underline{\nabla}_v \chi &= \frac{1}{2} m_\alpha \underline{\nabla}_v (\underline{v} \cdot \underline{v}) \\ &= m_\alpha (\underline{v} \cdot \underline{\nabla}_v) \underline{v} \\ &= m_\alpha \underline{v} \end{aligned} \quad (5.2)$$

Therefore, the terms in the left-hand side of the general transport equation (2.13) become

$$\frac{\partial}{\partial t} (n_{\alpha} \langle \chi_{\alpha} \rangle) = \frac{3}{2} \frac{\partial p_{\alpha}}{\partial t} + \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_{\alpha} u_{\alpha}^2 \right) \quad (5.3a)$$

$$\nabla \cdot (n_{\alpha} \langle \chi \underline{v} \rangle_{\alpha}) = \nabla \cdot \left[\frac{1}{2} \rho_{\alpha} \langle (\underline{v} \cdot \underline{v}) \underline{v} \rangle_{\alpha} \right] \quad (5.3b)$$

$$- n_{\alpha} \langle (F/m_{\alpha}) \cdot \underline{v} \rangle_{\alpha} = - n_{\alpha} \langle \underline{F} \cdot \underline{v} \rangle_{\alpha} \quad (5.3c)$$

Adding these terms, results in the following *energy conservation equation*

$$\begin{aligned} \frac{3}{2} \frac{\partial p_{\alpha}}{\partial t} + \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_{\alpha} u_{\alpha}^2 \right) + \frac{1}{2} \nabla \cdot \left[\rho_{\alpha} \langle (\underline{v} \cdot \underline{v}) \underline{v} \rangle_{\alpha} \right] \\ - n_{\alpha} \langle \underline{F} \cdot \underline{v} \rangle_{\alpha} = M_{\alpha} \end{aligned} \quad (5.4)$$

where M_{α} represents the time rate of change of energy per unit volume, due to collisions,

$$M_{\alpha} \equiv \frac{1}{2} m_{\alpha} \int_{\mathbf{v}} \mathbf{v}^2 \left(\frac{\delta f_{\alpha}}{\delta t} \right)_{\text{coll}} d^3\mathbf{v} = \left[\frac{\delta (\rho_{\alpha} \langle \mathbf{v}^2 \rangle_{\alpha} / 2)}{\delta t} \right]_{\text{coll}} \quad (5.5)$$

The energy equation (5.4) can be written in an *alternative form* as follows. Consider, initially, the third term in the left-hand side of (5.4). Taking $\underline{\mathbf{v}} = \underline{\mathbf{c}}_{\alpha} + \underline{\mathbf{u}}_{\alpha}$, the quantity $\langle (\underline{\mathbf{v}} \cdot \underline{\mathbf{v}}) \underline{\mathbf{v}} \rangle_{\alpha}$ can be expanded as

$$\begin{aligned} \langle [(\underline{\mathbf{u}}_{\alpha} + \underline{\mathbf{c}}_{\alpha}) \cdot (\underline{\mathbf{u}}_{\alpha} + \underline{\mathbf{c}}_{\alpha})] (\underline{\mathbf{u}}_{\alpha} + \underline{\mathbf{c}}_{\alpha}) \rangle &= \\ &= \langle (\underline{\mathbf{u}}_{\alpha}^2 + 2 \underline{\mathbf{u}}_{\alpha} \cdot \underline{\mathbf{c}}_{\alpha} + \underline{\mathbf{c}}_{\alpha}^2) (\underline{\mathbf{u}}_{\alpha} + \underline{\mathbf{c}}_{\alpha}) \rangle \\ &= \underline{\mathbf{u}}_{\alpha}^2 \underline{\mathbf{u}}_{\alpha} + \langle \underline{\mathbf{c}}_{\alpha}^2 \rangle \underline{\mathbf{u}}_{\alpha} + 2 \langle \underline{\mathbf{c}}_{\alpha} \underline{\mathbf{c}}_{\alpha} \rangle \cdot \underline{\mathbf{u}}_{\alpha} + \langle \underline{\mathbf{c}}_{\alpha}^2 \underline{\mathbf{c}}_{\alpha} \rangle \end{aligned} \quad (5.6)$$

The term $\rho_{\alpha} \langle \underline{\mathbf{c}}_{\alpha} \underline{\mathbf{c}}_{\alpha} \rangle$ represents the kinetic pressure dyad $\underline{\mathbf{p}}_{\alpha}$ and $\rho_{\alpha} \langle \underline{\mathbf{c}}_{\alpha}^2 \underline{\mathbf{c}}_{\alpha} \rangle / 2$ is the heat flux vector $\underline{\mathbf{q}}_{\alpha}$, defined in Chapter 6. We have also seen that $\rho_{\alpha} \langle \underline{\mathbf{c}}_{\alpha}^2 \rangle / 2 = 3p_{\alpha} / 2$. Therefore,

$$\begin{aligned}
 \frac{1}{2} \underline{\nabla} \cdot \left[\rho_{\alpha} < (\underline{v} \cdot \underline{v}) \underline{v} >_{\alpha} \right] &= \underline{\nabla} \cdot \left(\frac{1}{2} \rho_{\alpha} u_{\alpha}^2 \underline{u}_{\alpha} + \frac{3}{2} p_{\alpha} \underline{u}_{\alpha} + \underline{p}_{\alpha} \cdot \underline{u}_{\alpha} + \underline{q}_{\alpha} \right) \\
 &= \underline{\nabla} \cdot \left(\frac{1}{2} \rho_{\alpha} u_{\alpha}^2 \underline{u}_{\alpha} \right) + \frac{3}{2} p_{\alpha} \underline{\nabla} \cdot \underline{u}_{\alpha} + \\
 &+ \frac{3}{2} \underline{u}_{\alpha} \cdot \underline{\nabla} p_{\alpha} + \underline{\nabla} \cdot (\underline{p}_{\alpha} \cdot \underline{u}_{\alpha}) + \\
 &+ \underline{\nabla} \cdot \underline{q}_{\alpha}
 \end{aligned} \tag{5.7}$$

Substituting this result into (5.4) and using the notation D/Dt for the total time derivative (4.7), we obtain

$$\begin{aligned}
 \frac{D}{Dt} \left(\frac{3}{2} p_{\alpha} \right) + \frac{3}{2} p_{\alpha} \underline{\nabla} \cdot \underline{u}_{\alpha} + \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_{\alpha} u_{\alpha}^2 \right) + \underline{\nabla} \cdot \left(\frac{1}{2} \rho_{\alpha} u_{\alpha}^2 \underline{u}_{\alpha} \right) + \\
 + \underline{\nabla} \cdot (\underline{p}_{\alpha} \cdot \underline{u}_{\alpha}) + \underline{\nabla} \cdot \underline{q}_{\alpha} - n_{\alpha} < \underline{F} \cdot \underline{v} >_{\alpha} = \dot{M}_{\alpha}
 \end{aligned} \tag{5.8}$$

The third and fourth terms in the left-hand side of this equation can be written as

$$\begin{aligned}
 & \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_{\alpha} \underline{u}_{\alpha} \cdot \underline{u}_{\alpha} \right) + \underline{\nabla} \cdot \left[\frac{1}{2} \rho_{\alpha} (\underline{u}_{\alpha} \cdot \underline{u}_{\alpha}) \underline{u}_{\alpha} \right] = \\
 & = \frac{1}{2} u_{\alpha}^2 \frac{\partial \rho_{\alpha}}{\partial t} + \rho_{\alpha} \underline{u}_{\alpha} \cdot \frac{\partial \underline{u}_{\alpha}}{\partial t} + \\
 & + \frac{1}{2} u_{\alpha}^2 \underline{\nabla} \cdot (\rho_{\alpha} \underline{u}_{\alpha}) + \rho_{\alpha} \underline{u}_{\alpha} \cdot \left[(\underline{u}_{\alpha} \cdot \underline{\nabla}) \underline{u}_{\alpha} \right] \\
 & = \frac{1}{2} u_{\alpha}^2 \left[\frac{\partial \rho_{\alpha}}{\partial t} + \underline{\nabla} \cdot (\rho_{\alpha} \underline{u}_{\alpha}) \right] + \rho_{\alpha} \underline{u}_{\alpha} \cdot \left(\frac{D \underline{u}_{\alpha}}{Dt} \right) \quad (5.9)
 \end{aligned}$$

Using the continuity equation (3.2) and the equation of motion (4.6), this last equation becomes

$$\frac{1}{2} u_{\alpha}^2 S_{\alpha} + n_{\alpha} \underline{u}_{\alpha} \cdot \langle \underline{F} \rangle_{\alpha} - \underline{u}_{\alpha} \cdot (\underline{\nabla} \cdot \underline{p}_{\alpha}) + \underline{u}_{\alpha} \cdot \underline{A}_{\alpha} = u_{\alpha}^2 S_{\alpha} \quad (5.10)$$

Taking this result back into Eq. (5.8) we obtain

$$\frac{D}{Dt} \left(-\frac{3}{2} p_{\alpha} \right) + \frac{3}{2} p_{\alpha} \underline{\nabla} \cdot \underline{u}_{\alpha} + \underline{\nabla} \cdot (\underline{p}_{\alpha} \cdot \underline{u}_{\alpha}) - \underline{u}_{\alpha} \cdot (\underline{\nabla} \cdot \underline{p}_{\alpha}) -$$

$$\begin{aligned}
 - n_{\alpha} \langle \underline{\tilde{F}} \cdot \underline{\tilde{v}} \rangle_{\alpha} + n_{\alpha} \underline{\tilde{u}}_{\alpha} \cdot \langle \underline{\tilde{F}} \rangle_{\alpha} + \underline{\tilde{\nabla}} \cdot \underline{\tilde{q}}_{\alpha} = \\
 = M_{\alpha} - \underline{\tilde{u}}_{\alpha} \cdot \underline{\tilde{A}}_{\alpha} + \frac{1}{2} u_{\alpha}^2 S_{\alpha}
 \end{aligned} \tag{5.11}$$

The third and fourth terms in the left-hand side of this equation can be combined in one single term,

$$\underline{\tilde{\nabla}} \cdot (\underline{\tilde{p}}_{\alpha} \cdot \underline{\tilde{u}}_{\alpha}) - \underline{\tilde{u}}_{\alpha} \cdot (\underline{\tilde{\nabla}} \cdot \underline{\tilde{p}}_{\alpha}) = (\underline{\tilde{p}}_{\alpha} \cdot \underline{\tilde{\nabla}}) \cdot \underline{\tilde{u}}_{\alpha} \tag{5.12}$$

as well as the fifth and the sixth terms, which give

$$- n_{\alpha} \langle \underline{\tilde{F}} \cdot \underline{\tilde{v}} \rangle_{\alpha} + n_{\alpha} \underline{\tilde{u}}_{\alpha} \cdot \langle \underline{\tilde{F}} \rangle_{\alpha} = - n_{\alpha} \langle \underline{\tilde{F}} \cdot \underline{\tilde{c}}_{\alpha} \rangle \tag{5.13}$$

since

$$\langle \underline{\tilde{F}} \cdot \underline{\tilde{v}} \rangle_{\alpha} = \langle \underline{\tilde{F}} \cdot (\underline{\tilde{u}}_{\alpha} + \underline{\tilde{c}}_{\alpha}) \rangle = \langle \underline{\tilde{F}} \rangle_{\alpha} \cdot \underline{\tilde{u}}_{\alpha} + \langle \underline{\tilde{F}} \cdot \underline{\tilde{c}}_{\alpha} \rangle \tag{5.14}$$

For a velocity-independent force Eq. (5.13) vanishes, since, in this case,

$$\langle \underline{\tilde{F}} \cdot \underline{\tilde{c}}_{\alpha} \rangle = \underline{\tilde{F}} \cdot \langle \underline{\tilde{c}}_{\alpha} \rangle = 0 \tag{5.15}$$

For the force due to a magnetic field $\underline{B}$, the only velocity-dependent force that we are interested here, Eq. (5.13) also vanishes,

$$\begin{aligned}
 \langle \underline{F} \cdot \underline{c}_\alpha \rangle &= q_\alpha \langle (\underline{v} \times \underline{B}) \cdot \underline{c}_\alpha \rangle \\
 &= q_\alpha (\underline{u}_\alpha \times \underline{B}) \cdot \langle \underline{c}_\alpha \rangle + q_\alpha \langle (\underline{c}_\alpha \times \underline{B}) \cdot \underline{c}_\alpha \rangle \\
 &= 0
 \end{aligned} \tag{5.16}$$

where both terms vanish since $\langle \underline{c}_\alpha \rangle = 0$ and $(\underline{c}_\alpha \times \underline{B})$ is normal to $\underline{c}_\alpha$. We obtain, finally, the following *alternative form* for the equation of *conservation of energy*

$$\begin{aligned}
 \frac{D}{Dt} \left(\frac{3}{2} p_\alpha \right) + \frac{3}{2} p_\alpha (\underline{\nabla} \cdot \underline{u}_\alpha) + (\underline{p}_\alpha \cdot \underline{\nabla}) \cdot \underline{u}_\alpha + \underline{\nabla} \cdot \underline{q}_\alpha &= \\
 = M_\alpha - \underline{u}_\alpha \cdot \underline{A}_\alpha + \frac{1}{2} u_\alpha^2 S_\alpha &
 \end{aligned} \tag{5.17}$$

5.2 - Physical interpretation

The physical interpretation of this equation is as follows. The first term in the left-hand side represents the total rate of change of the thermal energy density of the particles of type α , in a volume element moving with the mean velocity $\underline{u}_\alpha$. Note that the thermal energy density is given by $\rho_\alpha \langle c_\alpha^2 \rangle / 2 = 3p_\alpha / 2$.

The other terms of Eq. (5.17) contribute to some extent to this total rate of change of the thermal energy per unit volume. The second term in the left-hand side of (5.17) can be interpreted as the change in the thermal energy density due to particles entering the volume element with the mean velocity $\underline{u}_\alpha$. The third term is related to the work done on the unit volume by the kinetic pressure dyad acting on its surface, whereas the fourth term represents the change in the thermal energy density due to the heat flux. Finally, the terms in the right-hand side of (5.17) represent the rate of change in the thermal energy density, as a consequence of the collisional interactions between the particles of type α and the other species of particles. In the case of a fluid containing only one type of particles, the collision terms vanish, as indicated previously.

The first two terms in the energy equation may also be combined, making use of the continuity equation (3.2). Expanding the term $\underline{\nabla} \cdot (\rho_\alpha \underline{u}_\alpha)$, Eq (3.2) becomes

$$\left(\frac{\partial}{\partial t} + \underline{u}_\alpha \cdot \underline{\nabla} \right) \rho_\alpha + \rho_\alpha \underline{\nabla} \cdot \underline{u}_\alpha = S_\alpha \quad (5.18)$$

which gives

$$\underline{\nabla} \cdot \underline{u}_\alpha = - \frac{1}{\rho_\alpha} \frac{D\rho_\alpha}{Dt} + \frac{1}{\rho_\alpha} S_\alpha \quad (5.19)$$

Substituting this result into Eq. (5.17), taking $\rho_\alpha = n_\alpha m_\alpha$ and $p_\alpha = n_\alpha k T_\alpha$, yields the following alternative form for the energy equation in terms of the temperature T_α ,

$$\frac{3}{2} n_\alpha k \frac{DT_\alpha}{Dt} + (\underline{p}_\alpha \cdot \underline{\nabla}) \cdot \underline{u}_\alpha + \underline{\nabla} \cdot \underline{q}_\alpha = M_\alpha - \underline{u}_\alpha \cdot \underline{A}_\alpha + \left(\frac{1}{2} u_\alpha^2 - \frac{3}{2} \frac{kT_\alpha}{m_\alpha} \right) S_\alpha \quad (5.20)$$

5.3 - Simplifying approximations

Several simplifying approximations can be considered for the energy equation, depending on the situation of interest.

(a) When the collision terms vanish, or are negligible, and when the mean velocity $\underline{u}_\alpha$ is equal to zero, Eq. (5.20) reduces to a *diffusion* equation for T_α , if we take the heat flux vector as

$$\underline{q}_\alpha = - K \underline{\nabla} T_\alpha \quad (5.21)$$

where K denotes the thermal conductivity. Thus, in this case,

$$\frac{3}{2} n_\alpha k \frac{DT_\alpha}{Dt} = \underline{\nabla} \cdot (K \underline{\nabla} T_\alpha) \quad (5.22)$$

The coefficient of thermal conductivity K is related to the coefficient of viscosity of the fluid.

(b) Consider now a non-viscous fluid, in which the pressure dyad reduces to a scalar pressure without thermal conductivity, that is, $q_\alpha = 0$. If we admit also that the collision terms vanish, the energy equation (5.17) becomes

$$\frac{D}{Dt} \left(\frac{3}{2} p_\alpha \right) + \frac{3}{2} p_\alpha (\underline{\nabla} \cdot \underline{u}_\alpha) + p_\alpha (\underline{\nabla} \cdot \underline{u}_\alpha) = 0 \quad (5.23)$$

Substituting (5.19) for $(\underline{\nabla} \cdot \underline{u}_\alpha)$, with $S_\alpha = 0$, we obtain

$$\frac{3}{2} \frac{Dp_\alpha}{Dt} - \frac{5}{2} \frac{p_\alpha}{\rho_\alpha} \frac{D\rho_\alpha}{Dt} = 0 \quad (5.24)$$

from which results

$$\frac{Dp_\alpha}{p_\alpha} = \frac{5}{3} \frac{D\rho_\alpha}{\rho_\alpha} \quad (5.25)$$

Integrating this equation gives

$$\frac{p_\alpha}{p_0} = \left(\frac{\rho_\alpha}{\rho_0} \right)^{5/3} \quad (5.26)$$

where p_0 and ρ_0 are constants, that is,

$$p_{\alpha} \rho_{\alpha}^{-5/3} = \text{constant} \quad (5.29)$$

This is the *adiabatic energy equation* for a gas in which the ratio of the specific heats at constant pressure and at constant volume, γ , is equal to 5/3. We emphasize here that the energy equation reduces to this adiabatic equation only when the effects of viscosity, thermal conductivity and energy transfer due to collisions are neglected.

The parameter γ is related to the number of degrees of freedom, N , of a gas, by the condition

$$\gamma = \frac{2 + N}{N} \quad (5.30)$$

For particles that have no internal degrees of freedom, as for example in a monoatomic gas, where the only degrees of freedom are those associated with the three possible directions of translational motion, we have $N = 3$ and, therefore, $\gamma = 5/3$. Other degrees of freedom exist in the case of diatomic or poliatomic gases. The *adiabatic energy equation* often used in thermodynamics is

$$p \rho^{-\gamma} = \text{constant} \quad (5.31)$$

Differentiating this equation, yields

$$\rho^{-\gamma} dp - \gamma p \rho^{-(\gamma+1)} d\rho = 0 \quad (5.32)$$

or equivalently,

$$dp = \frac{\gamma p}{\rho} d\rho = V_S^2 d\rho \quad (5.33)$$

where we have defined

$$V_S = (\gamma p / \rho)^{1/2} = (\gamma k T / m)^{1/2} \quad (5.34)$$

which is the *adiabatic speed of sound* for the fluid.

(c) An equation which is also used in thermodynamics when the temperature is constant inside the fluid, is the *isothermal energy equation*. It can be easily obtained from the equation of state for an ideal gas, $p = n k T$. For an isothermal process ($T = \text{constant}$), we have

$$\begin{aligned} dp &= k T \, dn = \frac{p}{\rho} d\rho \\ &= V_T^2 d\rho \end{aligned} \quad (5.35)$$

where

$$V_T = (p / \rho)^{1/2} = (k T / m)^{1/2} \quad (5.36)$$

is the *isothermal speed of sound*.

6. THE COLD PLASMA MODEL

In the previous sections we have seen that the differential equations governing the temporal and spatial variation of the macroscopic variables can be obtained by taking moments of the Boltzmann equation. The macroscopic parameters are all moments of the distribution function $f_\alpha(\underline{r}, \underline{v}, t)$. The first four moments of the distribution function give us the number density n_α , the mean velocity $\underline{u}_\alpha$, the momentum flow dyad $\underline{p}_\alpha$, and the energy flow triad $\underline{E}_\alpha$. The first moment of the Boltzmann equation gives us the *continuity equation* which relates the number density n_α (or the mass density ρ_α) with the mean velocity $\underline{u}_\alpha$ for the particles of type α . In order to determine these two macroscopic variables we need two independent equations. Thus, we evaluate the second moment of the Boltzmann equation, which gives us the *equation of motion*, and which relates the mean velocity $\underline{u}_\alpha$, the number density n_α and the kinetic pressure dyad $\underline{p}_\alpha$. We find, therefore, that the set of transport equations derived from the Boltzmann equation always includes more variables than independent equations. This fact is clearly evident in the three transport equations derived in this chapter. The *energy equation*, besides the variables n_α , $\underline{u}_\alpha$ and $\underline{p}_\alpha$, also includes the vector of heat flow $\underline{q}_\alpha$. Any finite set of transport equations is, therefore, insufficient to form a *closed* system of equations. Consequently, it is necessary to introduce a scheme of approximation to eliminate some of the independent variables, or to express some of these variables in terms of the others. It is common, therefore, to

arbitrarily truncate the system of transport equations at some point in the hierarchy of moments of the Boltzmann equation, and consider some simplifying approximation for the highest moment of the distribution function appearing in the system.

The simplest closed system of transport equations is known as the *cold plasma model*, which encompasses only the equations of conservation of mass and momentum. The highest moment of the distribution function, appearing in the momentum equation, is the kinetic pressure dyad $\underline{p}_{\alpha}$ which, in this model, is taken equal to zero. This means that the effects due to the thermal motion of the particles and the force due to the divergence of the kinetic pressure dyad $\underline{p}_{\alpha}$ are neglected. For convenience, we collect here the two transport equations pertinent to the cold plasma model,

$$\frac{\partial \rho_{\alpha}}{\partial t} + \nabla \cdot (\rho_{\alpha} \underline{u}_{\alpha}) = S_{\alpha} \quad (6.1)$$

$$\rho_{\alpha} \frac{D \underline{u}_{\alpha}}{Dt} = n_{\alpha} q_{\alpha} (\underline{E} + \underline{u}_{\alpha} \times \underline{B}) + \rho_{\alpha} \underline{g} + \underline{A}_{\alpha} - \underline{u}_{\alpha} S_{\alpha} \quad (6.2)$$

In the absence of processes leading to production and loss of particles of type α (such as ionization and recombination), we have $S_{\alpha} = 0$. The expression normally used for the collision term for momentum transfer, $\underline{A}_{\alpha}$, is the one indicated in Eq. (4.11).

This model has been successfully applied in the investigation of the properties of small amplitude electromagnetic waves propagating in plasmas, with phase velocities much larger than the thermal velocity of the particles; this theory is known as *magnetoionic theory*.

7. THE WARM PLASMA MODEL

In this closed system of transport equations, the simplifying approximation is introduced in the equation of conservation of energy, in which we neglect the term involving the heat flux vector $\underline{q}_\alpha$. Thus, the approximation consists in taking $\underline{\nabla} \cdot \underline{q}_\alpha = 0$, which means that the processes occurring in the plasma are such that there is no thermal energy flux. This approximation is also called *adiabatic approximation*. Since the thermal conductivity is zero in this case, it follows that the plasma is non-viscous and, consequently, the non-diagonal terms of the kinetic pressure dyad $\underline{p}_\alpha$ are all equal to zero. Further, the diagonal terms of $\underline{p}_\alpha$ are assumed to be equal and the kinetic pressure dyad is replaced by a scalar pressure p_α . Thus, the term $\underline{\nabla} \cdot \underline{p}_\alpha$ degenerates to $\underline{\nabla} p_\alpha$.

The three macroscopic variables appearing in this case are the number density n_α , the mean velocity $\underline{u}_\alpha$ and the scalar

pressure p_α . The three transport equations pertaining to the warm plasma model are, therefore,

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \underline{u}_\alpha) = S_\alpha \quad (7.1)$$

$$\rho_\alpha \frac{D \underline{u}_\alpha}{Dt} = n_\alpha q_\alpha (\underline{E} + \underline{u}_\alpha \times \underline{B}) + \rho_\alpha \underline{g} - \nabla p_\alpha + \underline{A}_\alpha - \underline{u}_\alpha S_\alpha \quad (7.2)$$

$$\frac{D}{Dt} \left(\frac{3}{2} p_\alpha \right) + \frac{5}{2} p_\alpha (\nabla \cdot \underline{u}_\alpha) = M_\alpha - \underline{u}_\alpha \cdot \underline{A}_\alpha + \frac{1}{2} u_\alpha^2 S_\alpha \quad (7.3)$$

Considering the additional approximation that the change in energy, as a result of collisional interactions, is negligible, the energy equation (7.3) reduces to the following *adiabatic equation* (as shown in section 5),

$$p_\alpha \rho_\alpha^{-\gamma} = \text{constant} \quad (7.4)$$

Generally, the warm plasma model gives a more precise description of the behavior of plasma phenomena as compared to the cold plasma model.

In the most general cases, in which the plasma is not in a state of local equilibrium, and when heat flow and viscosity need to be taken into account, it is more convenient and simple to work directly with the phase space distribution function. In this case, the plasma is usually said to be *hot*. After determining the distribution function $f_{\alpha}(\underline{r}, \underline{v}, t)$, by solving the differential kinetic equation which governs the evolution of $f_{\alpha}(\underline{r}, \underline{v}, t)$ in phase space, for the specific problem under consideration, the macroscopic variables can be obtained according to the systematic method presented in Chapter 6.

PROBLEMS

8.1 - Consider the following simplified steady state equation of motion, for each species in a fluid plasma

$$0 = qn (\underline{E} + \underline{u} \times \underline{B}) - \underline{\nabla} p$$

where the electric ($\underline{E}$) and magnetic ($\underline{B}$) fields are uniform, but the number density (n) and the pressure (p) have a gradient. Taking the cross product of this equation with $\underline{B}$ show that, besides the $\underline{E} \times \underline{B}$ drift, there is also a *diamagnetic drift* given by

$$\underline{v}_D = - \frac{\underline{\nabla} p \times \underline{B}}{n q B^2}$$

Provide physical arguments to justify the physical reason for this fluid drift. Explain if there is any motion of the particle's guiding centers, associated with this fluid drift.

8.2 - (a) From Maxwell equations,

$$\underline{\nabla} \cdot \underline{E} = \rho_c / \epsilon_0$$

$$\underline{\nabla} \cdot \underline{H} = 0$$

$$\underline{\nabla} \times \underline{E} = - \mu_0 \frac{\partial \underline{H}}{\partial t}$$

$$\underline{\nabla} \times \underline{H} = \underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

where $\underline{E}$ and $\underline{H}$ denote the electric and magnetic fields in a plasma, ρ_c denotes the charge density nq , and $\underline{J}$ the current density $nq \underline{u}$, show that

$$\epsilon_0 \mu_0 \frac{\partial}{\partial t} (\underline{E} \times \underline{H}) = \underline{\nabla} \cdot \underline{T} - nq (\underline{E} + \underline{u} \times \underline{B}) \quad (A)$$

where $\epsilon_0 \mu_0 (\underline{E} \times \underline{H})$ is the electromagnetic momentum density, $\underline{T}$ is the electromagnetic stress dyad whose components are given by

$$T_{ij} = \epsilon_0 E_i E_j + \mu_0 H_i H_j - \frac{1}{2} \delta_{ij} (\epsilon_0 E^2 + \mu_0 H^2)$$

and δ_{ij} is the Kronecker delta.

(b) From Eq. (A) and the continuity equation

$$\frac{\partial n}{\partial t} + \underline{\nabla} \cdot (n \underline{u}) = 0$$

show that the momentum transport equation

$$nm \frac{D\underline{u}}{Dt} = nq (\underline{E} + \underline{u} \times \underline{B}) - \underline{\nabla} \cdot \underline{p}$$

can be written in the form

$$\frac{\partial \underline{G}}{\partial t} + \underline{\nabla} \cdot \underline{\pi} = 0$$

where $\underline{G}$ denotes the total momentum density

$$\underline{G} = n m \underline{u} + \epsilon_0 \mu_0 (\underline{E} \times \underline{H})$$

and $\underline{\pi}$ is the total momentum flux (rate of transport of momentum through unit area)

$$\underline{\pi} = n m \underline{u} \underline{u} + \underline{p} - \underline{T} = \underline{P} - \underline{T}$$

(c) Using Maxwell curl equations, show that the energy transport equation

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2} n m \langle c^2 \rangle \right) + \frac{\partial}{\partial t} \left(\frac{1}{2} n m u^2 \right) = - \underline{\nabla} \cdot \left(\frac{1}{2} n m \langle c^2 \rangle \underline{u} \right) - \\ - \underline{\nabla} \cdot \left(\frac{1}{2} n m u^2 \underline{u} \right) - \underline{\nabla} \cdot \underline{q} - \underline{\nabla} \cdot (\underline{p} \cdot \underline{u}) + \\ + n q \underline{u} \cdot \underline{E} \end{aligned}$$

[note that $\underline{u} \cdot (\underline{u} \times \underline{B}) = 0$] can be written in the form

$$\frac{\partial W}{\partial t} + \underline{\nabla} \cdot \underline{S} = 0$$

where W denotes the total energy density

$$W = \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 + \frac{1}{2} n m u^2 + \frac{1}{2} n m \langle c^2 \rangle$$

and $\underline{S}$ is the total energy flux (power per unit area)

$$\underline{S} = \underline{E} \times \underline{H} + \underline{p} \cdot \underline{u} + \frac{1}{2} n m \underline{u} (\langle c^2 \rangle + u^2) + \underline{q}$$

8.3 - In order to investigate the effect of the collision term (4.11) in the macroscopic fluid motion, consider a uniform mixture of different fluids (all spatial derivatives vanish), with no external forces, so that the equation of motion for the α species reduces to

$$\frac{d \underline{u}_{\alpha}}{dt} = - \sum_{\beta} v_{\alpha\beta} (\underline{u}_{\alpha} - \underline{u}_{\beta})$$

Solve this equation to determine $\underline{u}_{\alpha}(t)$, for a two-fluid mixture and a three-fluid mixture. (In the case of the three-fluid mixture, it is better to use Laplace transforms). Notice that, at equilibrium (when $du_{\alpha}/dt = 0$), the velocities of all species must be the same.

8.4 - Consider a uniform mixture of different fluids (all spatial derivatives vanish), with no external forces, such that the equation of motion for the α species becomes

$$\frac{d \underline{u}_{\alpha}}{dt} = - \sum_{\beta} v_{\alpha\beta} (\underline{u}_{\alpha} - \underline{u}_{\beta})$$

(a) Show that the time rate of change of the total fluid kinetic energy density, W_k , is given by

$$\frac{d W_k}{dt} = - \frac{1}{2} \sum_{\alpha, \beta} \rho_{\alpha} v_{\alpha\beta} (\underline{u}_{\alpha} - \underline{u}_{\beta})^2$$

where

$$W_k = \sum_{\alpha} \frac{1}{2} \rho_{\alpha} u_{\alpha}^2$$

(b) Consider now the total fluid thermal energy density,

$$W_T = \sum_{\alpha} \frac{3}{2} n_{\alpha} k T_{\alpha}$$

If the energy equation for a homogeneous fluid mixture, with no external forces, is

$$\frac{d T_{\alpha}}{dt} = - \sum_{\beta} \frac{2m_{\alpha} v_{\alpha\beta}}{(m_{\alpha} + m_{\beta})} \left[(T_{\alpha} - T_{\beta}) - \frac{m_{\beta}}{3k} (\underline{u}_{\alpha} - \underline{u}_{\beta})^2 \right]$$

then, show that the time rate of change of W_T is given by

$$\frac{d W_T}{dt} = \frac{1}{2} \sum_{\alpha, \beta} \rho_{\alpha} v_{\alpha\beta} (\underline{u}_{\alpha} - \underline{u}_{\beta})^2$$

Thus, the total thermal energy density, W_T , increases at exactly the same rate as the total kinetic energy density, W_k , decreases.

Hint: for any function which is summed over two indices, the result is unchanged if we interchange the indices, so that

$$\sum_{\alpha, \beta} f_{\alpha\beta} = \sum_{\alpha, \beta} f_{\beta\alpha}$$

or

$$\sum_{\alpha, \beta} f_{\alpha\beta} = \frac{1}{2} \sum_{\alpha, \beta} (f_{\alpha\beta} + f_{\beta\alpha})$$

8.5 - Explain the reason why there is no term containing the magnetic flux density, $\underline{B}$, in the energy equation (5.17).

8.6 - Derive the following *general transport equation*, similar to (2.13), for the case when the quantity χ depends on $\underline{r}$, $\underline{v}$ and t :

$$\begin{aligned} \frac{\partial}{\partial t} (n_{\alpha} \langle \chi \rangle_{\alpha}) &= n_{\alpha} \langle \partial \chi / \partial t \rangle_{\alpha} + \underline{\nabla} \cdot (n_{\alpha} \langle \chi \underline{v} \rangle_{\alpha}) - \\ &- n_{\alpha} \langle (\underline{v} \cdot \underline{\nabla}) \chi \rangle_{\alpha} - n_{\alpha} \langle \underline{a} \cdot \underline{\nabla}_{\underline{v}} \chi \rangle_{\alpha} = \\ &= \left[\frac{\delta}{\delta t} (n_{\alpha} \langle \chi \rangle_{\alpha}) \right]_{\text{coll}} \end{aligned}$$

8.7 - Consider the general transport equation of Problem 8.6, and let the property $\chi(\underline{r}, \underline{v}, t)$ be the random flux of thermal kinetic energy, that is, $m_{\alpha} c_{\alpha}^2 c_{\alpha} / 2$, where $\underline{c}_{\alpha} = \underline{v} - \underline{u}_{\alpha}(\underline{r}, t)$. Show that [with $\underline{a} = (q_{\alpha} / m_{\alpha}) (\underline{E} + \underline{v} \times \underline{B})$],

$$\begin{aligned} \frac{\partial}{\partial t} (n_{\alpha} \langle \chi \rangle_{\alpha}) &= \frac{\partial}{\partial t} q_{\alpha} \\ n_{\alpha} \langle \partial \chi / \partial t \rangle_{\alpha} &= - \left(\frac{\partial u_{\alpha}}{\partial t} \right) \cdot \left(\underline{p}_{\alpha} + \frac{3}{2} p_{\alpha} \underline{1} \right) \\ \underline{\nabla} \cdot (n_{\alpha} \langle \chi \underline{v} \rangle_{\alpha}) &= \underline{\nabla} \cdot \left(\frac{1}{2} \rho_{\alpha} \langle c_{\alpha}^2 \underline{c}_{\alpha} \underline{c}_{\alpha} \rangle + \underline{u}_{\alpha} q_{\alpha} \right) \end{aligned}$$

$$n_{\alpha} \langle (\underline{v} \cdot \underline{\nabla}) \chi \rangle_{\alpha} = - (\underline{Q}_{\alpha} \cdot \underline{\nabla}) \cdot \underline{u}_{\alpha} - (\underline{q}_{\alpha} \cdot \underline{\nabla}) \underline{u}_{\alpha} - \left[(\underline{u}_{\alpha} \cdot \underline{\nabla}) \underline{u}_{\alpha} \right] .$$

$$\cdot \left(\underline{p}_{\alpha} + \frac{3}{2} \underline{p}_{\alpha} \underline{1} \right)$$

$$n_{\alpha} \langle \underline{a} \cdot \underline{\nabla} \chi \rangle_{\alpha} = \rho_{\alpha} \langle \underline{a} \cdot (\underline{c}_{\alpha} \underline{c}_{\alpha} + \frac{1}{2} c_{\alpha}^2 \underline{1}) \rangle =$$

$$= \frac{q_{\alpha}}{m_{\alpha}} \left[(\underline{E} + \underline{u}_{\alpha} \times \underline{B}) \cdot \left(\underline{p}_{\alpha} + \frac{3}{2} \underline{p}_{\alpha} \underline{1} \right) + \underline{q}_{\alpha} \times \underline{B} \right]$$

Using these results in the general transport equation, derive the following equation, known as the *heat flow equation*,

$$\frac{\partial}{\partial t} \underline{q}_{\alpha} - \frac{1}{\rho_{\alpha}} (\underline{\nabla} \cdot \underline{p}_{\alpha}) \cdot \left(\underline{p}_{\alpha} + \frac{3}{2} \underline{p}_{\alpha} \underline{1} \right) + \underline{\nabla} \cdot \left(\frac{1}{2} \rho_{\alpha} \langle c_{\alpha}^2 \underline{c}_{\alpha} \underline{c}_{\alpha} \rangle + \right.$$

$$\left. + \underline{u}_{\alpha} \underline{q}_{\alpha} \right) + (\underline{Q}_{\alpha} \cdot \underline{\nabla}) \cdot \underline{u}_{\alpha} + (\underline{q}_{\alpha} \cdot \underline{\nabla}) \underline{u}_{\alpha} - \frac{q_{\alpha}}{m_{\alpha}} (\underline{q}_{\alpha} \times \underline{B}) =$$

$$= \left(\frac{\delta}{\delta t} \underline{q}_{\alpha} \right)_{\text{coll}}$$

8.8 - In the general transport equation of Problem 8.6, consider that the property $\chi(\underline{r}, \underline{v}, t)$ is the random momentum flux, that is, $m_\alpha c_{\alpha j} c_{\alpha k}$. Show that

$$\frac{\partial}{\partial t} (n_\alpha \langle \chi \rangle_\alpha) = \frac{\partial}{\partial t} (p_{\alpha jk})$$

$$n_\alpha \langle \partial \chi / \partial t \rangle_\alpha = 0$$

$$\frac{\partial}{\partial x_i} (n_\alpha \langle \chi v_i \rangle_\alpha) = \frac{\partial}{\partial x_i} (Q_{\alpha ijk} + u_{\alpha i} p_{\alpha jk})$$

$$n_\alpha \langle v_i (\partial \chi / \partial v_i) \rangle_\alpha = -p_{\alpha ij} \frac{\partial u_{\alpha k}}{\partial x_i} - p_{\alpha ik} \frac{\partial u_{\alpha j}}{\partial x_i}$$

$$n_\alpha \langle a_i (\partial \chi / \partial v_i) \rangle_\alpha = \rho_\alpha \langle a_j c_{\alpha k} + a_k c_{\alpha j} \rangle$$

where the summation convention on repeated indices is being used. Plug these results in the general transport equation to derive the following equation, known as the *viscous stress equation*,

$$\frac{\partial}{\partial t} (p_{\alpha jk}) + \frac{\partial}{\partial x_i} (Q_{\alpha ijk} + u_{\alpha i} p_{\alpha jk}) + p_{\alpha ij} \frac{\partial u_{\alpha k}}{\partial x_i} + p_{\alpha ik} \frac{\partial u_{\alpha j}}{\partial x_i} -$$

$$- \rho_\alpha \langle a_j c_{\alpha k} + a_k c_{\alpha j} \rangle = \left[\frac{\delta}{\delta t} (p_{\alpha jk}) \right]_{\text{coll}}$$

8.9 - Verify that the energy conservation equation (for the random kinetic energy $m_\alpha c_\alpha^2/2$) can be obtained from the viscous stress equation (see Problem 8.8) by letting $j = k$, and summing over k .

8.10 - From the heat flow equation, derived in Problem 8.7, obtain the following simplified equation for heat flow in a stationary ($\underline{u} = 0$) electron gas

$$\frac{5}{2} p_e \underline{\nabla} \left(\frac{p_e}{\rho_e} \right) + \omega_{ce} (\underline{q}_e \times \underline{B}) = \left(\frac{\delta}{\delta t} \underline{q}_e \right)_{\text{coll}}$$

State all the assumptions necessary to obtain this result.

8.11 - Using the relaxation model (or Krook collision model) for the collision term, $(\delta f_\alpha / \delta t) = -\nu(f_\alpha - f_{0\alpha})$, and the ideal gas law $p_e = n_e k T_e$, show that the heat flow equation of Problem 8.10 becomes

$$\underline{q}_e + \frac{\omega_{ce}}{\nu} (\underline{q}_e \times \underline{B}) = -K_0 \underline{\nabla} T_e$$

where $K_0 = 5kp_e/2m_e\nu$ is the thermal conductivity.