1. Classification <i>INPE-COM.4/RPE</i> <i>C.D.U.: 533.9</i>		2. Period	4. Distribution Criterion
3. Key Words (selected by the author) <i>PLASMA OSCILLATIONS</i> <i>DEBYE SHIELDING</i> <i>PLASMA SHEATH</i>		internal ☐	external ☒
5. Report No. <i>INPE-1502-RPE/042</i>	6. Date <i>July, 1979</i>	7. Revised by <i>Alicia L.C. de Gonzalez</i>	
8. Title and Sub-title <i>SOME BASIC PLASMA PHENOMENA</i>		9. Authorized by <i>Nelson de Jesus Parada</i> Director	
10. Sector <i>DCE/DGA/GIO</i>	Code <i>30.371</i>	11. No. of Copies <i>13</i>	
12. Authorship <i>J. A. Bittencourt</i>		14. No. of Pages <i>58</i>	
13. Signature of first author <i>Bittencourt</i>		15. Price	
16. Summary/Notes <i>This chapter describes three basic phenomena in plasmas: the electron plasma oscillations, the Debye shielding and the plasma sheath. The electron plasma oscillations and the Debye shielding problem are analysed using the macroscopic transport equations for each particle species in the plasma. To illustrate the use of the phase space distribution function, the problem of the Debye potential is also analysed in terms of the Vlasov equation. The physical mechanism of formation of the plasma sheath, near a boundary wall in the plasma, and an approximate quantitative description of the inner structure of the sheath, are also presented. Finally, as an application of the properties of the plasma sheath, it is presented an analysis of the Langmuir plasma probe, which is widely used to measure plasma properties.</i>			
17. Remarks			

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CHAPTER 11

SOME BASIC PLASMA PHENOMENA

1. ELECTRON PLASMA OSCILLATIONS

One of the fundamental properties of a plasma is its tendency to maintain electric charge neutrality on a macroscopic scale, under equilibrium conditions. When this macroscopic charge neutrality is disturbed such as to temporarily produce a significant imbalance of charge, large Coulomb forces come into play which tend to restore the macroscopic charge neutrality. Since these Coulomb forces cannot be naturally sustained in the plasma, it breaks into high frequency electron plasma oscillations, which enable the plasma to maintain on the average its electrical neutrality.

As a simple example, consider a small spherical region inside the plasma and suppose that a perturbation in the form of an excess of negative charge is introduced in this small spherical plasma region. Due to this negative charge, a macroscopic electric field is produced. Because of spherical symmetry, this field is radial and points towards the center of the spherical plasma region (Fig. 1), forcing the electrons to move radially outward. After a small time interval, since the electrons gain kinetic energy in the course of their motion, more electrons leave the spherical plasma region (due

to their inertia) than is necessary to resume the state of electrical neutrality. An excess of positive charge results, therefore, inside the spherical plasma region and the electric field gets reserved, causing the electrons to move inward now. This movement of electrons inward and outward of the spherical plasma region continues periodically, resulting in electron plasma oscillations. In this way the plasma maintains its macroscopic neutrality on the average, since the total charge inside the spherical region averaged over one period of these oscillations is zero. The frequency of these oscillations is usually very high and since the ions, in view of their heavy mass, are unable to follow the rapidity of these oscillations, their motion is often neglected.

Fig. 1 - The radial electric field $\underline{E}$ produced by a spherical distribution of negative charge (a) forces the electrons to move radially outward, whereas the field produced by a spherical distribution of positive charge (b) forces the electrons to move radially inward.

To study the characteristics of the electron plasma oscillations we can use the cold plasma model, in which the thermal motion of the particles and the pressure gradient force are not taken into account. We neglect the motion of the ions and assume that the perturbation in the electron density is very small, that is,

$$n_e(\underline{r}, t) = n_0 + n'_e(\underline{r}, t) \quad (1.1)$$

where n_0 is a constant number density and $|n'_e| \ll n_0$. Similarly, we assume that the electric field produced, $\underline{E}(\underline{r}, t)$, and the average electron velocity, $\underline{u}_e(\underline{r}, t)$, are first-order perturbations, so that the linearized equations can be used. The linearized equations of conservation of the number of particles and of momentum, become, respectively,

$$\frac{\partial n'_e}{\partial t} + n_0 \nabla \cdot \underline{u}_e = 0 \quad (1.2)$$

$$\frac{\partial \underline{u}_e}{\partial t} = - \frac{e}{m_e} \underline{E} \quad (1.3)$$

In the momentum equation, we have assumed that the rate of loss of momentum from the electron gas is zero, and we have neglected the magnetic force term, since the magnetic field $\underline{B}$ due to the motion of the electrons is a first-order perturbation, so that the product $\underline{u}_e \times \underline{B}$

gives a second order term. Denoting the ion charge by e , the total charge density becomes

$$\rho_c(\underline{r}, t) = -e [n_0 + n'_e(\underline{r}, t)] + en_0 = -en'_e(\underline{r}, t) \quad (1.4)$$

where we have considered the ion density to be constant and uniform, and equal to n_0 . Therefore,

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho_c}{\epsilon_0} = - \left(\frac{e}{\epsilon_0} \right) n'_e \quad (1.5)$$

Taking the divergence of Eq. (1.3) and using Eq. (1.2) to substitute for $\underline{\nabla} \cdot \underline{u}_e$, we obtain

$$\frac{\partial^2 n'_e}{\partial t^2} - \left(\frac{en_0}{m_e} \right) \underline{\nabla} \cdot \underline{E} = 0 \quad (1.6)$$

Using Maxwell divergence equation (1.5), we can write Eq. (1.6) in the form

$$\frac{\partial^2 n'_e}{\partial t^2} + \omega_{pe}^2 n'_e = 0 \quad (1.7)$$

where

$$\omega_{pe} = (n_0 e^2 / m_e \epsilon_0)^{1/2} \quad (1.8)$$

is called the *electron plasma frequency*. From Eq. (1.7) we see that $n'_e(\underline{r}, t)$ varies harmonically in time at the electron plasma frequency,

$$n'_e(\underline{r}, t) = n'_e(\underline{r}) \exp(-i\omega_{pe} t) \quad (1.9)$$

In fact, all first-order perturbations also have a harmonic time variation at the plasma frequency ω_{pe} . To verify this statement, it is convenient to start with the assumption that all first order quantities vary harmonically in time, as $\exp(-i\omega t)$. Eqs. (1.2) and (1.3) become, in this case,

$$n'_e = -\left(\frac{i}{\omega}\right) n_0 \underline{\nabla} \cdot \underline{u}_e \quad (1.10)$$

$$\underline{u}_e = -\left(\frac{ie}{\omega m_e}\right) \underline{E} \quad (1.11)$$

which can be combined into

$$n'_e = -\frac{en_0}{\omega^2 m_e} \underline{\nabla} \cdot \underline{E} \quad (1.12)$$

Substituting this expression for n'_e into Maxwell divergence equation (1.5), yields

$$(1. - \frac{\omega_{pe}^2}{\omega^2}) \nabla \cdot \underline{\underline{E}} = 0 \quad (1.13)$$

which shows that a non-trivial solution gives $\omega = \omega_{pe}$. Therefore, all the perturbations vary harmonically in time at the electron plasma frequency. Further, the perturbation number density and all other field variables change with time in phase, at every point in space, and there is no change in phase from point to point, implying in the absence of wave propagation. The oscillations are, therefore, *stationary*. From Eq. (1.11) we see that the electron velocity and, therefore, the motion of the electrons is in the same direction as the electric field, which shows that the oscillations are also *longitudinal*.

The electron plasma oscillations are also *electrostatic* in character. In order to show this aspect of the oscillations, consider Maxwell curl equations with a harmonic time variation,

$$\nabla \times \underline{\underline{E}} = i \omega \underline{\underline{B}} \quad (1.14)$$

$$\nabla \times \underline{\underline{B}} = \mu_0 (\underline{\underline{J}} - i \omega \epsilon_0 \underline{\underline{E}}) \quad (1.15)$$

The electric current density is given by

$$\underline{\underline{J}} = -en_0 \underline{\underline{u}}_e = \left(\frac{i n_0 e^2}{\omega m_e} \right) \underline{\underline{E}} \quad (1.16)$$

where we have used Eq. (1.11) for $\underline{u}_e$. Eq. (1.15) becomes, therefore,

$$\underline{\nabla} \times \underline{B} = -i \omega \mu_0 \epsilon_0 \epsilon_r \underline{E} \quad (1.17)$$

where we have defined a relative permittivity as

$$\epsilon_r = 1 - \omega_{pe}^2 / \omega^2 \quad (1.18)$$

For the electron plasma oscillations, we have $\omega = \omega_{pe}$, so that $\epsilon_r = 0$, and Eq. (1.17) reduces to

$$\underline{\nabla} \times \underline{B} = 0 \quad (1.19)$$

Since the curl of the gradient of any scalar function vanishes identically, we may write, as a consequence of (1.19),

$$\underline{B} = \underline{\nabla} \psi \quad (1.20)$$

where ψ is a magnetic scalar potential. Substituting (1.20) into (1.14) and taking the divergence of both sides we obtain Laplace equation

$$\underline{\nabla} \cdot (\underline{\nabla} \psi) = \nabla^2 \psi = 0 \quad (1.21)$$

since the divergence of the curl of any vector function vanishes identically. The only solution of this equation, which is not singular and finite at infinity, is $\psi = \text{constant}$, so that

$$\underline{B} = 0 \quad (1.22)$$

which shows that the oscillations are electrostatic in character. In summary, the electron plasma oscillations (at the electron plasma frequency ω_{pe}) are *stationary, longitudinal and electrostatic*. Characteristic values of the electron plasma frequency for various laboratory and cosmic plasmas are given in Fig. 2 of Chapter 1.

2. THE DEBYE SHIELDING PROBLEM

To examine the mechanism by which the plasma strives to shield its interior from a disturbing electric field, consider a plasma whose equilibrium state is perturbed by an electric field due to an external charged particle. For that matter, this electric field may also be considered to be due to one of the charged particles inside the plasma, isolated for observation. For definiteness, we assume this *test particle* to have a positive charge $+Q$, and choose a spherical coordinate system whose origin coincides with the position of the test particle. We are interested in determining the electrostatic potential $\phi(\underline{r})$ that is established near the test charge Q , due to the *combined* effects of the test charge and the distribution of charged particles surrounding it. Since the positive test charge Q attracts the negatively charged particles and repels the positively charged ones, the number densities of the electrons $n_e(\underline{r})$ and of the ions $n_i(\underline{r})$ will be slightly different near the origin (test particle), whereas at large distances from the origin the electrostatic potential vanishes so that $n_e = n_i = n_0$. Since this is a steady state problem under the action of

a conservative electric field, we have

$$\underline{E}(\underline{r}) = - \underline{\nabla} \phi(\underline{r}) \quad (2.1)$$

and from Eq. (7.5.16) it follows that

$$n_e(\underline{r}) = n_0 \exp [e \phi(\underline{r}) / kT] \quad (2.2)$$

$$n_i(\underline{r}) = n_0 \exp [-e \phi(\underline{r}) / kT] \quad (2.3)$$

where we have assumed that the electrons and ions (of charge e) have the same temperature T .

The total electric charge density $\rho_c(\underline{r})$, including the test charge Q , can be expressed as

$$\rho_c(\underline{r}) = -e [n_e(\underline{r}) - n_i(\underline{r})] + Q \delta(\underline{r}) \quad (2.4)$$

where $\delta(\underline{r})$ denotes the Dirac delta function. Using Eqs. (2.2) and (2.3), we obtain

$$\begin{aligned} \rho_c(\underline{r}) = -e n_0 \left\{ \exp [e \phi(\underline{r}) / kT] - \exp [-e \phi(\underline{r}) / kT] \right\} + \\ + Q \delta(\underline{r}) \end{aligned} \quad (2.5)$$

Substituting (2.1) and (2.5) into the following Maxwell equation

$$\underline{\nabla} \cdot \underline{E}(\underline{r}) = \rho_c(\underline{r}) / \epsilon_0 \quad (2.6)$$

gives the differential equation

$$\begin{aligned} \nabla^2 \phi(\underline{r}) - \frac{e n_0}{\epsilon_0} \left\{ \exp \left[e \phi(\underline{r}) / kT \right] - \exp \left[- e \phi(\underline{r}) / kT \right] \right\} = \\ = - \frac{Q}{\epsilon_0} \delta(\underline{r}) \end{aligned} \quad (2.7)$$

which permits the evaluation of the electrostatic potential $\phi(\underline{r})$.

We assume now that the perturbing electrostatic potential is weak so that the electrostatic potential energy is much less than the mean thermal energy, that is,

$$e \phi(\underline{r}) \ll kT \quad (2.8)$$

Under this condition we can use the approximation

$$\exp \left[\pm e \phi(\underline{r}) / kT \right] = 1 \pm e \phi(\underline{r}) \quad (2.9)$$

Therefore, Eq. (2.7) simplifies to

$$\nabla^2 \phi(\underline{r}) - \frac{2}{\lambda_D^2} \phi(\underline{r}) = - \frac{Q}{\epsilon_0} \delta(\underline{r}) \quad (2.10)$$

where we have used the notation

$$\lambda_D = \left(\frac{\epsilon_0 kT}{n_0 e^2} \right)^{1/2} = \frac{1}{\omega_{pe}} \left(\frac{kT}{m_e} \right)^{1/2} \quad (2.11)$$

Since the problem has spherical symmetry, the electrostatic potential depends only on the radial distance r measured from the position of the test particle, and depends on the direction of $\underline{r}$. Thus, using spherical coordinates, Eq. (2.10) can be written (for $r \neq 0$) as

$$\left(\frac{1}{r^2}\right) \frac{d}{dr} \left[r^2 \frac{d}{dr} \phi(r) \right] - \frac{2}{\lambda_D^2} \phi(r) = 0; \quad (r \neq 0) \quad (2.12)$$

We note now that for an isolated particle of charge $+Q$, in free space, the electric field is directed radially outward and is given by

$$\underline{E}(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \underline{\hat{r}} \quad (2.13)$$

so that the electrostatic Coulomb potential $\phi_c(r)$, due to this isolated charged particle in free space, is

$$\phi_c(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \quad (2.14)$$

Since in the very close proximity of the test particle the electrostatic potential should be the same as that for an isolated particle in free space, we seek the solution of Eq. (2.12) in the form

$$\phi(r) = \phi_c(r) \cdot F(r) = \frac{Q}{4\pi\epsilon_0} \cdot \frac{F(r)}{r} \quad (2.15)$$

where the function $F(r)$ must be such that $F(r) \rightarrow 1$ when $r \rightarrow 0$. Furthermore, the electrostatic potential $\phi(r)$ is required to vanish at infinity, that is, $\phi(r) \rightarrow 0$ when $r \rightarrow \infty$.

Substituting Eq. (2.15) into Eq. (2.12) yields the following differential equation for $F(r)$

$$\frac{d^2 F(r)}{dr^2} = \frac{2}{\lambda_D^2} F(r) \quad (2.16)$$

This simple differential equation has the solution

$$F(r) = A \exp(\sqrt{2} r / \lambda_D) + B \exp(-\sqrt{2} r / \lambda_D) \quad (2.17)$$

To fulfill the condition that the potential $\phi(r)$ vanishes for large values of r , we must take $A = 0$. Also, the condition that the function $F(r)$ tends to one when r tends to zero requires that we take $B = 1$. Therefore, the solution of Eq. (2.12) is

$$\begin{aligned}\phi(r) &= \phi_c(r) \exp(-\sqrt{2}r/\lambda_D) \\ &= \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r} \exp(-\sqrt{2}r/\lambda_D)\end{aligned}\quad (2.18)$$

This result is commonly known as the *Debye potential*, since the nonrigorous derivation presented here was first given by Debye and Hückel in their theory of electrolytes. It shows that the electrostatic potential $\phi(r)$ becomes much less than the ordinary Coulomb potential once r exceeds the distance λ_D , called the *Debye length*. This is shown in Fig. 2: Hence, we can say, in a crude way, that a charged particle

Fig. 2 - Electrostatic Coulomb potential $\phi_c(r)$ and Debye potential $\phi(r)$ as a function of distance r from the test charge Q . ($\phi_0 = Q\sqrt{2}/4\pi\epsilon_0\lambda_D$).

in a plasma interacts effectively only with particles situated at distances less than one Debye length away, and has a negligible influence on particles lying at distances greater than one Debye length. Thus, the charged particles inside the Debye sphere of a test particle shields the major part of the plasma from the electric field of the test particle, this shielding occurring over a distance of the order of λ_D .

The charge Q of the test particle is neutralized by the resultant distribution of the charged particles in the plasma surrounding the test particle. From Eqs. (2.5) and (2.9), we obtain for the charge density

$$\rho_c(\underline{r}) = -2n_0 e^2 \phi(\underline{r}) / kT + Q \delta(\underline{r}) \quad (2.19)$$

Substituting $\phi(\underline{r})$ by the Debye potential (2.18), we obtain

$$\rho_c(\underline{r}) = -\frac{Q}{2\pi r \lambda_D^2} \exp(-\sqrt{2} r / \lambda_D) + Q \delta(\underline{r}) \quad (2.20)$$

To calculate the total charge q_t we integrate Eq. (2.20) throughout all space

$$\begin{aligned}
 q_t &= \iiint \rho_c(\underline{r}) d^3r \\
 &= - \frac{Q}{2\pi \lambda_D^2} \int_0^\infty \frac{1}{r} \exp(-\sqrt{2} r / \lambda_D) 4\pi r^2 dr + \\
 &+ Q \iiint \delta(\underline{r}) d^3r
 \end{aligned} \tag{2.21}$$

The first integral gives $-Q$, whereas the second one is equal to $+Q$. Therefore, we obtain

$$q_t = 0 \tag{2.2}$$

Since the charge of the test particles was explicitly included in the expression for $\rho_c(\underline{r})$. The principal contribution to the first integral in (2.21) comes from the plasma particles lying in the very close neighborhood of the test particle, since the integrand falls off exponentially with increasing values of r . Thus, in effect, the neutralization of the test particle takes place on account of the charged particles inside the Debye sphere. From Eqs. (2.2) and (2.3) we see that in neighborhood of the test particles the number density of the electrons is larger than that of the ions, on account of the fact that the

positive test particle attracts the electrons and repels the ions. Therefore, in the close proximity of the test particle there is an imbalance of charge and, consequently, an electric field is present. We have seen that the shielding of this electric field is effectively completed over a distance of the order of λ_D . Thus, in order that the plasma be considered macroscopically neutral, it is necessary that its typical characteristic dimension L be much greater than λ_D . This quantitative *criterion* for the definition of a plasma has been previously discussed in Chapter 1.

An important point to be noted in the result (2.18) for the Debye potential is that, for r tending to zero, the Debye potential becomes very large and the assumption $e \phi(r) \ll kT$ is unlikely to be fulfilled. To verify the validity of this approximation, and consequently of the result (2.18), note that using Eq. (2.18) with $Q = e$ we have

$$\frac{e\phi}{kT} = \frac{e^2 \exp(-\sqrt{2} r / \lambda_D)}{4\pi \epsilon_0 r kT} = \frac{\lambda_D}{3 N_D} \frac{\exp(-\sqrt{2} r / \lambda_D)}{r} \quad (2.23)$$

where

$$N_D = \frac{4}{3} \pi \lambda_D^3 n_e \quad (2.24)$$

is the number of electrons inside a Debye sphere. Since N_D is very large for virtually all plasmas, it is evident that the ratio given in Eq. (2.23) is much less than one, except when r is less than λ_D/N_D . Therefore, the Debye potential (2.18) is consistent with the approximation $e\phi \ll kT$ used to derive it, if we restrict attention to distances from the test particle greater than λ_D/N_D .

As a final point, we note that in the derivation of the Debye potential, which appears extensively in the literature, it is usual to ignore the motion of the ions and assume for their number density a constant value n_0 , equal to the unperturbed electron number density. In this case, the factor of 2 disappears from Eq. (2.10), and the expression for the Debye potential becomes

$$\phi(r) = \frac{1}{4\pi \epsilon_0} \frac{Q}{r} \exp(-r / \lambda_D) \quad (2.25)$$

3. DEBYE SHIELDING USING THE VLASOV EQUATION

As an example of the use of the Vlasov equation we reconsider in this section the Debye potential problem. As before, we suppose that a test charged particle $+Q$ is placed at the origin of a spherical coordinate system inside the plasma. We want to determine the steady state distribution functions for the electrons and the ions, f_e and f_i , and the electrostatic potential $\phi(\underline{r})$ near the test particle Q .

The steady state Vlasov equations for the electrons and the ions, considering only the electric field $\underline{E}$ in the Lorentz force term, are

$$\underline{v} \cdot \nabla f_e + \frac{e}{m_e} (\nabla \phi) \cdot \nabla_v f_e = 0 \quad (3.1)$$

$$\underline{v} \cdot \nabla f_i - \frac{e}{m_i} (\nabla \phi) \cdot \nabla_v f_i = 0 \quad (3.2)$$

where we have taken

$$\underline{E} = - \underline{\nabla} \phi \quad (3.3)$$

Since

$$n_{\alpha}(\underline{r}) = \int_{\underline{v}} f_{\alpha}(\underline{r}, \underline{v}) d^3v \quad (3.4)$$

the total charge density, including the test particle, can be expressed as

$$\rho_c(\underline{r}) = -e \int_{\underline{v}} (f_e - f_i) d^3v + Q \delta(\underline{r}) \quad (3.5)$$

Using Eqs. (3.3) and (3.5) in Maxwell equation (2.6), we obtain Poisson equation for this case

$$\nabla^2 \phi - \frac{e}{\epsilon_0} \int_{\underline{v}} (f_e - f_i) d^3v = - \frac{Q}{\epsilon_0} \delta(\underline{r}) \quad (3.6)$$

Eqs. (3.1), (3.2) and (3.6) constitute three equations to be solved simultaneously to determine f_e , f_i and ϕ .

We have seen in section 5, of Chapter 7, that the solution of the Vlasov equations (3.1) and (3.2) can be expressed

in terms of the Maxwellian distribution function and the Boltzmann factor, as

$$f_{\alpha}(\underline{r}, v) = f_{o\alpha}(v) \exp \left[-q_{\alpha} \phi(\underline{r}) / kT \right] \quad (3.7)$$

When the electrostatic potential $\phi(\underline{r})$ vanishes, this distribution function goes into the Maxwellian distribution $f_{o\alpha}(v)$, with zero drift velocity. Substituting (3.7), with $\alpha = e, i$, into (3.6), yields

$$\begin{aligned} \nabla^2 \phi &= - \frac{e}{\epsilon_0} \left[\exp(e\phi / kT) \int_v f_{oe}(v) d^3v - \exp(-e\phi / kT) \int_v f_{oi}(v) d^3v \right] = \\ &= - \frac{Q}{\epsilon_0} \delta(\underline{r}) \end{aligned} \quad (3.8)$$

If we denote the number density of the electrons and of the ions, under the equilibrium conditions when ϕ vanishes, by n_0 , that is,

$$n_0 = \int_v f_{o\alpha}(v) d^3v \quad \alpha = e, i \quad (3.9)$$

then Eq. (3.8) becomes

$$\nabla^2 \phi = - \frac{en_0}{\epsilon_0} \left[\exp(e\phi / kT) - \exp(-e\phi / kT) \right] = - \frac{Q}{\epsilon_0} \delta(\underline{r}) \quad (3.10)$$

This equation is identical to (2.7). We obtain, therefore, the same result as before for the Debye potential.

4. PLASMA SHEATH

When a material body is immersed in a plasma, the body acquires a net negative charge and, therefore, a negative potential with respect to the potential inside the plasma. In the region near the wall of the body there is a boundary layer, known as the *plasma sheath*, in which the number densities of the electrons and the ions are not equal. Inside the plasma sheath the potential increases monotonically from a negative value on the wall to the value corresponding to the unperturbed plasma. The thickness of the plasma sheath, where departures from macroscopic electric neutrality occur, is found to be of the order of a Debye length.

A satisfactory mathematical treatment of this phenomenon is quite involved and cannot be presented here. However, the basic physical factors responsible for the formation of the plasma sheath are fairly simple to understand. In this section, we will set up the problem mathematically and obtain some approximate results. The problem is strongly dependent on the particular geometry under consideration. For simplicity, we will assume that the wall bounding the plasma is an infinite plane surface at $x = 0$, with the plasma in the region $x > 0$, and that there are no variations with respect to the coordinates y and z .

4.1 - Physical mechanism

We begin with a descriptive account of the physical mechanism responsible for the formation of the plasma sheath. The charged particles in the plasma that strike the wall in virtue of their random thermal motions are for the most part lost to the plasma. The ions generally recombine at the wall and return to the plasma as neutral particles, whereas the electrons may either recombine there or they may enter the conduction band if the surface is a metal. We have seen in section 4, of Chapter 7, that the *random* particle flux, that is, the number of particles that hit the surface per unit area and per unit time, from one side only, for the case of an isotropic velocity distribution function, is given by [see Eqs. (7.4.35) and (7.4.18)]

$$\Gamma_{\alpha} = n_{\alpha} \langle v \rangle_{\alpha} / 4 \quad (4.1)$$

where $\langle v \rangle_{\alpha}$ is the average particle speed for the α species. For the Maxwell-Boltzmann velocity distribution function we have found that [see Eq. (7.4.20)]

$$\langle v \rangle_{\alpha} = \left(\frac{8k T_{\alpha}}{\pi m_{\alpha}} \right)^{1/2} \quad (4.2)$$

so that a typical value for the random particle flux Γ_{α} , in this case, is

$$\Gamma_{\alpha} = n_{\alpha} \left(\frac{k T_{\alpha}}{2 \pi m_{\alpha}} \right)^{1/2} \quad (4.3)$$

It is evident from this result that, if initially the number densities of the electrons and the ions are equal, then the random particle flux for the electrons, Γ_e , greatly exceeds that for the ions, Γ_i , since in general $(T_e/m_e)^{1/2}$ is much larger than $(T_i/m_i)^{1/2}$. For the least heavy ion, hydrogen, for example, m_i is greater than m_e by a factor of 1836. Therefore, the wall in contact with the plasma rapidly accumulates a negative charge, since initially more electrons reach the wall than positive ions. The negative potential associated with this negative charge repels the electrons and attracts the ions and, consequently, the electron flux diminishes and the ion flux increases. Eventually, the negative potential at the wall becomes large enough in magnitude to equalize the rate at which electrons and ions hit the surface. At this floating negative potential, the wall and the plasma reach a dynamical equilibrium such that the net current at the wall is zero.

4.2 - Electric potential on the wall

To estimate the value of the potential on the wall after the plasma sheath has been established, consider a steady state situation and let the electric potential $\phi(x)$ at the wall ($x = 0$) be given by

$$\phi (0) = \phi_w \quad (4.4)$$

Let us choose the reference potential inside the plasma, at a very large distance from the wall, equal to zero,

$$\phi (\infty) = 0 \quad (4.5)$$

The electrons and the ions are assumed to be in thermodynamic equilibrium at the same temperature T , under the action of the conservative electric field associated with the negative potential on the wall. At $x = \infty$ the plasma is not perturbed and the number density of the electrons and ions are equal, which we denote by n_0 . According to the results of section 5, of Chapter 7, the number densities of the electrons and the ions can be expressed as

$$n_e (r) = n_0 \exp [e \phi (r) / kT] \quad (4.6)$$

$$n_i (r) = n_0 \exp [-e \phi (r) / kT] \quad (4.7)$$

It is important to note at this point that Eqs. (4.6) and (4.7) do not take into account the drift velocity of the particles towards the wall. Since the electrons and the ions impinging on the surface of the wall are for the most part lost to the plasma, there must be a steady

flux of both species towards the wall to replenish this loss of charged particles. Despite this inadequacy, we shall still use these equations to obtain an approximate expression for the potential on the wall, and afterwards, in order to study the inner structure of the plasma sheath, we will take into account the particle drift in an approximation manner, using the hydrodynamic equations.

One of the boundary conditions of the problem is that, under equilibrium conditions, there must be no charge build up at the wall ($x=0$), so that

$$\Gamma_e(0) = \Gamma_i(0) \quad (4.8)$$

Using Eq. (4.3) together with Eqs. (4.6) and (4.7) we have, therefore,

$$\left(\frac{1}{m_e}\right)^{1/2} \exp(e\phi_w/kT) = \left(\frac{1}{m_i}\right)^{1/2} \exp(-e\phi_w/kT) \quad (4.9)$$

which may be written as

$$\exp(-2e\phi_w/kT) = (m_i/m_e)^{1/2} \quad (4.10)$$

Taking the natural logarithm of both sides, and solving for the potential on the wall, we obtain

$$\phi_w = -\frac{kT}{4e} \ln\left(\frac{m_i}{m_e}\right) \quad (4.11)$$

Other more accurate methods of calculating the potential on the wall yield results which, for $T_e = T_i$, agree qualitatively with the one given in Eq. (4.11), despite the inadequacy of Eqs. (4.6) and (4.7) which neglect the drift velocity of the particles towards the wall.

Note from Eq. (4.11) that the magnitude of the potential energy near the wall $|e \phi_w|$ is of the same order as the average thermal energy kT of the particles in the plasma, since

$$\frac{|e \phi_w|}{kT} = \frac{1}{4} \ln \left(\frac{m_i}{m_e} \right) \quad (4.12)$$

For a hydrogen ion, for example, $|e \phi_w| / kT$ is approximately equal to 2, whereas for heavier ions it may be close to 3.

4.3 - Inner structure of the plasma sheath

To investigate the inner structure of the plasma sheath we use the equations of conservation of mass and momentum for the electrons and ions. We assume steady state conditions with spatial dependence only on the x -direction. The equation of conservation of mass becomes, therefore,

$$\frac{d}{dx} (n_\alpha u_\alpha) = n_\alpha \frac{du_\alpha}{dx} + u_\alpha \frac{dn_\alpha}{dx} = 0 \quad (4.13)$$

In the equation of conservation of momentum, we neglect viscosity effects and approximate the kinetic pressure dyad by a scalar pressure, and use the ideal gas law $p_\alpha = n_\alpha k T_\alpha$. The collisional interactions are also neglected, since the thickness of the plasma sheath is much less than the mean free path for the plasma particles. With this assumptions and in the absence of a magnetic field, the equation of motion becomes (with $\underline{E} = -\underline{\nabla} \phi$, and $D/DT = \partial/\partial t + \underline{u}_\alpha \cdot \underline{\nabla} = u_\alpha d/dx$)

$$m_\alpha u_\alpha \frac{du_\alpha}{dx} = - \frac{kT}{n_\alpha} \frac{dn_\alpha}{dx} - q_\alpha \frac{d\phi(x)}{dx} \quad (4.14)$$

In order to simplify the analysis, we will make two approximations. The ratio of the magnitude of the term in the left-hand side of Eq. (4.14) to the first term in the right-hand side, is given by

$$\frac{\left| m_\alpha u_\alpha \frac{du_\alpha}{dx} \right|}{\left| \frac{kT}{n_\alpha} \frac{dn_\alpha}{dx} \right|} = \frac{m_\alpha u_\alpha^2}{kT} \quad (4.15)$$

where we have used, from Eq. (4.13),

$$u_\alpha \frac{dn_\alpha}{dx} = - n_\alpha \frac{du_\alpha}{dx} \quad (4.16)$$

The two approximations consist in neglecting the term in the left-hand side of Eq. (4.14) for the electrons, whereas for the ions we neglect the first term in the right-hand side of Eq. (4.14). Explicitly, we take for the electrons

$$\frac{kT}{n_e} \frac{dn_e}{dx} - e \frac{d\phi(x)}{dx} = 0 \quad (4.17)$$

and for the ions

$$m_i u_i \frac{du_i}{dx} + e \frac{d\phi(x)}{dx} = 0 \quad (4.18)$$

These two approximations are justified only if the thermal energy of the *electrons* is much *larger* than the kinetic energy due to the flow of electrons, and if the thermal energy of the *ions* is much *smaller* than the kinetic energy due to the flow of ions. Thus, we require, from Eq. (4.15), that

$$m_e u_e^2 \ll kT \ll m_i u_i^2 \quad (4.19)$$

in order to justify the approximations in Eqs. (4.17) and (4.18). We shall assume the condition (4.19) to be satisfied in the plasma sheath but it remains to be justified later in this section.

If we integrate Eq. (4.17), we obtain

$$e \phi (x) = kT \ln n_e (x) + (\text{constant}) \quad (4.20)$$

and using the condition that $n_e = n_0$ when $\phi = 0$, we find

$$n_e (x) = n_0 \exp [e \phi (x) / kT] \quad (4.21)$$

This result is identical to Eq. (4.6), which is not surprising, since the condition $m_e u_e^2 \ll kT$ is identical to neglecting the drift velocity of the electrons.

For the ions, we first integrate Eq. (4.13) to find

$$n_i (x) u_i (x) = C_1 \quad (4.22)$$

and then integrate Eq. (4.18) to obtain

$$e \phi (x) + \frac{1}{2} m_i u_i^2 (x) = C_2 \quad (4.23)$$

where C_1 and C_2 are constants. The boundary conditions require that, at $x = \infty$ we must have $\phi (\infty) = 0$, $n_i (\infty) = n_0$ and $u_i (\infty) = u_{0i}$. Thus,

$$C_1 = n_0 u_{0i} \quad C_2 = \frac{1}{2} m_i u_{0i}^2 \quad (4.24)$$

and using these results in Eqs. (4.22) and (4.23), we find

$$n_i(x) u_i(x) = n_o u_{oi} \quad (4.25)$$

$$e \phi(x) + \frac{1}{2} m_i u_i^2(x) = \frac{1}{2} m_i u_{oi}^2 \quad (4.26)$$

These two equations can be combined to eliminate $u_i(x)$ and solved for $n_i(x)$, giving

$$n_i(x) = n_o \left[1 - \frac{2 e \phi(x)}{m_i u_{oi}^2} \right]^{-1/2} \quad (4.27)$$

This expression for $n_i(x)$ is substantially different from the one given in Eq. (4.7) and this difference is due to the importance of the drift velocity of the ions. We now find that, since $\phi(x) < 0$ in the sheath, $n_i(x)$ decreases slowly towards the wall, rather than increasing as predicted by Eq. (4.7). Physically, this behavior is due to the fact that the negative potential on the wall causes $u_i(x)$ to increase as the ions approach the wall and since the flux $n_i(x) u_i(x)$ must stay constant, in virtue of Eq. (4.25), it turns out that $n_i(x)$ must decrease according to Eq. (4.27). This behavior is illustrated schematically in Fig. 3.

To obtain the differential equation satisfied by the electrostatic potential $\phi(x)$, we substitute Eqs. (4.21) and (4.27) into Poisson's equation

$$\nabla^2 \phi = \frac{e}{\epsilon_o} (n_e - n_i) \quad (4.28)$$

Fig. 3 - Diagram showing the variation of the electrostatic potential $\phi(x)$ and the number densities n_e and n_i inside the plasma sheath near an infinite plane wall.

to obtain

$$\frac{d^2\phi}{dx^2} = \frac{n_0 e}{\epsilon_0} \left[\exp\left(\frac{e\phi}{kT}\right) - \left(1 - \frac{2e\phi}{m_i u_{0i}^2}\right)^{-1/2} \right] \quad (4.29)$$

In this equation, the drift velocity u_{0i} , far away from the wall, still needs to be determined. This equation is nonlinear and in order to facilitate its analytical solution, we need to make one more approximation. Since we have seen that $|e\phi|$ ranges from zero in the plasma to a value of the order of kT on the wall, and since we have also assumed that $m_i u_{0i}^2$ is larger than kT , we will restrict our

attention to the region near the plasma edge of the sheath and assume further that $|e\phi|$ is small compared to both kT and $m_i u_{oi}^2$. Thus, in the region near the edge of the sheath adjacent to the plasma, we can expand the terms in the right-hand side of Eq. (4.29), for $(e\phi/kT) \ll 1$ and $(e\phi/m_i u_{oi}^2) \ll 1$, as

$$\exp\left(\frac{e\phi}{kT}\right) \approx 1 + \frac{e\phi}{kT} \quad (4.30)$$

$$\left(1 - \frac{2e\phi}{m_i u_{oi}^2}\right)^{-1/2} \approx 1 + \frac{e\phi}{m_i u_{oi}^2} \quad (4.31)$$

with the result that Eq. (4.29) reduces to

$$\frac{d^2 \phi}{dx^2} \approx \frac{\phi}{\chi^2} \quad (4.32)$$

where

$$\chi^2 = \lambda_D^2 \left(1 - \frac{kT}{m_i u_{oi}^2}\right)^{-1} \quad (4.33)$$

The solution of Eq. (4.32), with the boundary condition that $\phi(\infty) = 0$, is.

$$\phi(x) = A e^{-x/\chi} \quad (4.34)$$

where A is a constant. Since we have assumed that $kT \ll m_i u_{oi}^2$, it follows from Eq. (4.33) that X is real and approximately equal to λ_D . Therefore, we find that $\phi(x)$ increases exponentially (since A must be negative) as we move from inside the sheath into the plasma and goes to zero at very large distances from the wall. Since $X \approx \lambda_D$, this increase effectively takes place within a distance of the order of a Debye length. This solution for $\phi(x)$ is strictly valid only near the plasma edge of the sheath, but if it is continued to apply throughout the plasma sheath, we can impose the boundary condition on the wall, that is, $\phi(0) = \phi_w$, which requires that $A = \phi_w$.

If kT were greater than $m_i u_{oi}^2$, then X would be imaginary and the electric potential would be an oscillating function of distance near the wall. The condition

$$kT < m_i u_{oi}^2 \quad (4.35)$$

must, therefore, be satisfied for the formation of a plasma sheath; it is known as the *Bohm criterion*.

It is not a trivial matter to determine the potential on the wall using the hydrodynamic equations. All the approximate methods that have been suggested give results which, for $T_e = T_i$, agree reasonably well with the approximate value for ϕ_w given in Eq. (4.11). Furthermore, there is no consistent way to determine the ion drift velocity, u_{oi} , at $x = \infty$, toward the wall, but an approximate estimate

can be obtained as follows. Since the ion flux must be constant, from Eq. (4.25) we can equate the ion flux $n_0 u_{oi}$ at $x = \infty$ to its value $\Gamma_i(0)$ at the wall. Using Eq. (4.3) with $n_i(r)$ as given by Eq. (4.7) evaluated at the wall, we find

$$u_{oi} = \left(\frac{kT}{2\pi m_i} \right)^{1/2} \exp(-e\phi_w/kT) \quad (4.36)$$

Similarly, for the electrons we can equate the electron flux $n_0 u_{oe}$ at $x = \infty$ to its value $\Gamma_e(0)$ at the wall. From Eq. (4.3), with $n_e(r)$ given by Eq. (4.6), evaluated at the wall, we find

$$u_{oe} = \left(\frac{kT}{2\pi m_e} \right)^{1/2} \exp(e\phi_w/kT) \quad (4.37)$$

and using Eq. (4.9) we see that

$$u_{oe} = u_{oi} \quad (4.38)$$

To verify the validity of the approximations indicated in Eq. (4.19), we note first that, from the mass conservation equation, the particle flux $n_\alpha(x) u_\alpha(x)$ must be constant for all x and equal to the value $n_0 u_{o\alpha}$. From Eq. (4.21) we see that the *minimum* value of $n_e(x)$ is $n_0 \exp(e\phi_w/kT)$ since ϕ_w is negative. Therefore,

$$u_e = \frac{n_0}{n_e} u_{oe} \leq u_{oe} \exp(-e\phi_w/kT) \quad (4.39)$$

and using Eq. (4.37), we get

$$u_e \leq \left(\frac{kT}{2\pi m_e} \right)^{1/2} \quad (4.40)$$

or

$$\frac{kT}{m_e u_e^2} \geq 2\pi \quad (4.41)$$

in agreement with assumption (4.19). Similarly, from Eq. (4.27), the *maximum* value of $n_i(x)$ is n_0 , so that

$$u_i = \frac{n_0}{n_i} u_{oi} \geq u_{oi} \quad (4.42)$$

and using Eq. (4.36), we obtain

$$u_i \geq \left(\frac{kT}{2\pi m_i} \right)^{1/2} \exp(-e\phi_w/kT) \quad (4.43)$$

or

$$\frac{kT}{m_i u_i^2} \leq 2\pi \exp(2e\phi_w/kT) \approx 0.1 \quad (4.44)$$

where the result in the right has been obtained substituting ϕ_w by the value given in Eq. (4.11). Therefore, in view of Eqs. (4.41) and (4.44),

the thermal energy of the electrons is seen to be greater than the kinetic energy associated with their motion towards the wall, whereas for the ions the opposite situation is verified. From Eq. (4.44) we can verify that the Bohm criterion, for the formation of the plasma sheath, is also satisfied.

Although the quantitative aspects of the discussion presented here are very approximate, it provides, however, a satisfactory qualitative picture of the sheath.

5. THE PLASMA PROBE

The plasma probe is a device that has been widely used to measure the temperature and density of the particles in a plasma, both in the laboratory and in space vehicles. The electrostatic probe was developed by Langmuir and Mott-Smith, and the physical mechanism of its operation can be well explained using the theory of plasma sheaths presented in the previous section. A conducting probe, or electrode, is immersed in a plasma and the current that flows through it is measured for various potentials applied to the probe. The temperature and number density of the electrons can then be obtained from the characteristic of the resulting current-potential curve. When the surface of the probe is plane, the current-potential curve has a shape like that illustrated in Fig. 4.

The probe, when inserted in the plasma, is surrounded by a plasma sheath which shields the major portion of the plasma from

the disturbing field of the probe. The thickness of the sheath is of the order of a Debye length. When no current flows through the electrode, it is at the negative floating potential ϕ_w , which is the wall potential discussed in the previous section. Under these conditions, the number of electrons reaching the probe per unit time is equal to the number of positive ions reaching the probe per unit time. We assume the current to be positive when it flows in the direction away from the probe. The current associated with the electrons is directed away from the probe and, therefore, is

Fig. 4 - Characteristics of the current-potential curve of an electrostatic probe immersed in a plasma. The floating potential of the probe with reference to the plasma potential is denoted by ϕ_w

considered positive. Consequently, the electric current associated with the flow of ions is negative. Under equilibrium conditions, there is no net current flowing through the probe and its potential is the floating potential ϕ_w . When the potential is made more negative than ϕ_w , the electric current associated with the flow of electrons is reduced due to the increased repulsive force imposed by the probe electric field on the electrons. As the potential is made more negative, the contribution to the electric current arising from the electrons will eventually become negligible and the total electric current asymptotically approaches a constant negative value, corresponding to the electric current density J_i associated with only the flow of ions. The ions that reach the edge of the plasma sheath fall into the potential well and their current is practically unaffected if the potential is made even more negative. On the other hand, when the probe potential is increased from the negative value ϕ_w , more electrons reach the probe than ions per unit time due to the decrease in the repulsion force on the electrons, and the net electric current becomes positive. When the electric potential is zero, that is, when the probe is at the same potential as the plasma, there is no electric field near the electrode and, since the average thermal velocity of the electrons is much greater than that of the ions, the current density due to the flow of electrons J_{e0} (for $\phi = 0$) is much greater than the ion current density. If the potential is made sufficiently positive, a situation arises in which the current associated with the ions becomes negligible, but all the electrons that reach the edge of the sheath are collected by the probe. The

electron current density reaches a fairly constant value for sufficiently high positive values of ϕ . This plateau region in the current-potential curve of the probe is called the region of *saturation* of the electron current. For higher positive values of ϕ there are complications in the current-potential characteristic of the electrode due to the occurrence of another phenomena.

An approximate expression for the magnitude of the electric current density due to the flow of electrons, away from the region of saturation, can be obtained from Eq. (4.6) as

$$J_e = J_{e0} \exp (e \phi / kT_e) \quad (5.1)$$

where J_{e0} is the electric current density due to the flow of electrons, when the electric potential is zero. Since, for $\phi = 0$, we have $\Gamma_e = n_e \langle v \rangle_e / 4$, and using Eq. (4.2) for the average electron speed, we obtain,

$$J_{e0} = en_e \left(\frac{kT_e}{2\pi m_e} \right)^{1/2} \quad (5.2)$$

where n_e is the electron number density in the unperturbed region of the plasma. Note that, when ϕ is negative, the ions reaching the edge of the sheath continue to fall into the negative potential of the probe and, hence, the ion current density is a constant J_i in the negative potential region. Thus, we can express the probe current density, in the region where $\phi < 0$, as

$$J_p = J_{e0} \exp (e \phi / kT_e) - J_i \quad (\text{for } \phi < 0) \quad (5.3)$$

From this equation we can deduce the result

$$T_e = \frac{e}{k} \left\{ \frac{d}{d\phi} \left[\ln (J_p + J_i) \right] \right\}^{-1} \quad (5.4)$$

This expression can be used to determine the electron temperature as follows. First, the electrode is made sufficiently negative with reference to the plasma potential, so that the current that flows through the probe is due to the ions only. The measurement of this current gives directly the value J_i . Then, the current-potential characteristic of the probe is measured and a plot of $\ln (J_p + J_i)$ as a function of ϕ is made. This curve has a straight-line section corresponding to the probe potential less than the plasma potential, and the slope of this straight line gives the value of

$$\frac{d}{d\phi} \left[\ln (J_p + J_i) \right] \quad (5.5)$$

which, when substituted in Eq. (5.4), gives the electron temperature in the plasma.

After the electron temperature T_e has been determined, we can evaluate the electron number density from Eq. (5.2), which can be written as,

$$n_e = \frac{J_{e0}}{e} \left(\frac{2 \pi m_e}{k T_e} \right)^{1/2} \quad (5.6)$$

The value of J_{e0} is determined by measuring the probe current corresponding to the plateau (electron saturation) region of the current-potential characteristic of the probe.

PROBLEMS

11.1-Consider a stationary plasma (electrons and one type of ions) under steady state conditions at a uniform temperature T_0 , when perturbed by a point charge $+Q$ placed at the origin of a coordinate system. Using the collisionless hydrodynamic equation for the electrons and ions

$$\rho_\alpha \frac{D\mathbf{u}_\alpha}{Dt} = n_\alpha q_\alpha \mathbf{E} - \nabla p_\alpha \quad ; \quad (\alpha = e, i)$$

with the ideal gas law $p_\alpha = n_\alpha k T_0$, and Poisson equation

$$\nabla^2 \phi(\mathbf{r}) = -\rho_c(\mathbf{r})/\epsilon_0$$

obtain the following differential equation

$$\nabla^2 \phi(\mathbf{r}) - \frac{2}{\lambda_D^2} \phi(\mathbf{r}) = -\frac{Q}{\epsilon_0} \delta(\mathbf{r})$$

for the Debye potential $\phi(\mathbf{r})$. Assume that the number density of each species can be expressed as $n_\alpha = n_0 + n'_\alpha$, where n_0 is constant and $|n'_\alpha| \ll n_0$. What are the approximations necessary to obtain this result?

11.2 - Analyse the Debye potential problem considering only the motion of the electrons (ions stay immobile), and show that, in this case, the differential equation for the electric potential $\phi(\underline{r})$ is

$$\nabla^2 \phi(\underline{r}) - \frac{1}{\lambda_D^2} \phi(\underline{r}) = - \frac{Q}{\epsilon_0} \delta(\underline{r})$$

11.3 - When the macroscopic neutrality of a plasma is instantaneously perturbed by external means, the electrons react in a such a way as to give rise to oscillations at the electron plasma frequency $\omega_{pe} = (n_0 e^2 / m_e \epsilon_0)^{1/2}$. Consider these oscillations, but including the motion of the ions. Show that, in this case, the natural frequency of oscillation of the net charge density is given by

$$\omega = (\omega_{pe}^2 + \omega_{pi}^2)^{1/2}$$

where $\omega_{pi} = (n_0 e^2 / m_i \epsilon_0)^{1/2}$. Use the linearized equations of continuity and of momentum for each species, and Poisson equation, considering only the electric force due to the internal charge separation.

11.4 - Evaluate the negative electrostatic potential ϕ_w which appears on an infinite plane wall immersed in a plasma consisting of electrons of charge $-e$ and ions of charge Ze , under steady state conditions. Denote the electron and ion temperatures by T_e and T_i , respectively.

11.5 - Deduce an expression for the Debye potential for a test particle of charge $+Q$ immersed in a plasma consisting of electrons (charge $-e$) and ions of charge Ze , the temperature of the electrons and the ions being T_e and T_i , respectively. Show that, if $T_e \gg T_i$, then the Debye length is governed by the ion temperature T_i .

11.6 - Using the following expressions for the electron and ion number densities

$$n_e(r) = n_0 \exp [e \phi(r)/kT]$$

$$n_i(r) = n_0 \exp [-e \phi(r)/kT]$$

in the plasma sheath region formed between an infinite plane and a semi-infinite plasma, deduce the differential equation satisfied by the electric potential $\phi(x)$ in the plasma sheath.

Show that this differential equation can be written in the form

$$\frac{d^2 F}{d\zeta^2} = \sinh (F)$$

where $F = e\phi/kT$ and $\zeta = \sqrt{2} x/\lambda_D$. Assuming that at $x = \infty$ we have $n_e = n_i = n_0$, $F = 0$ and $dF/d\zeta = 0$, show that

$$F(\zeta) = 4 \tanh^{-1} \left\{ \exp \left[- (\zeta - \zeta_0) \right] \right\}$$

where ζ_0 is a constant. Denoting the potential at the wall by ϕ_w and assuming that $e\phi/kT \ll 1$, show that

$$\phi(x) = \phi_w \exp \left(\frac{\sqrt{2} x}{\lambda_D} \right)$$

with

$$\phi_w = \frac{4kT}{e} \exp (\zeta_0)$$

11.7 - For the plasma sheath region formed in the vicinity of a plane wall immersed in a plasma, assume that the ions at the plasma

edge of the sheath can be described by a shifted Maxwellian distribution function

$$f_i(v) = n_o \left(\frac{m_i}{2\pi kT} \right)^{3/2} \exp \left[- \frac{m_i (\underline{v} - \underline{u}_o)^2}{2 kT} \right]$$

with drift velocity $\underline{u}_o = u_o \hat{x}$. Prove that the ion flux Γ_{ix} , at the edge of the sheath, is given by

$$\Gamma_{ix} = n_o \left(\frac{kT}{2\pi m_i} \right)^{1/2} \left\{ \exp(-y^2) + y\sqrt{\pi} [1 + \text{erf}(y)] \right\}$$

where $y = u_o (m_i/2kT)^{1/2}$ and $\text{erf}(y)$ is the error function, defined by

$$\text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y \exp(-s^2) ds$$

Calculate $d\Gamma_{ix}/dy$. Note that the error function vanishes for $y = 0$, increases monotonically as y increases and tends asymptotically to unity as $y \rightarrow \infty$. Also note that

$$\frac{d}{dy} \text{erf}(y) = \frac{2}{\sqrt{\pi}} \exp(-y^2)$$

11.8 - From an experimental current-potential curve of a Langmuir probe of area A immersed in a plasma, such as shown in Fig. P11.1, where the electric potentials are measured with respect to a fixed reference potential, explain how you can determine J_i , J_{eo} , the space potential ϕ_s , the floating potential ϕ_f with respect to ϕ_s (note that $\phi_f - \phi_s = \phi_w$), T_e and n_e .

Fig. P11.1

11.9 - The Langmuir plasma probe has been widely used in satellites to measure space plasma properties. In one valuable technique, circuits are arranged that measure directly $dI_p/d\phi$ and $d^2I_p/d\phi^2$, where $I_p = J_p A$ and A is the probe's area. Use Eq. (5.3) to show that

$$\frac{(d I_p/d\phi)}{(d^2 I_p/d^2\phi)} = \frac{kT_e}{e}$$

which gives directly the electron temperature T_e . Next, show that J_{eo} can be calculated from T_e and $dI_p/d\phi$ at a known value of ϕ , according to

$$J_{eo} = \frac{kT_e}{Ae} \exp(-e\phi/kT_e) \left(\frac{d I_p}{d\phi} \right)$$

The electron density n can then be calculated from Eq. (5.6).

11.10 - An electron gas (Lorentz gas) in a background of stationary ions, is acted upon by a weak, externally applied electric field $\underline{E}$, under steady state conditions. Using the Boltzmann equation for the electrons, with the relaxation model for the collision term (with a constant collision frequency ν),

$$\left(\frac{\delta f}{\delta t} \right)_{coll} = -\nu (f - f_o)$$

and considering the adiabatic case for which

$$n(\underline{r}) [T(\underline{r})]^{-3/2} = \text{constant}$$

show that the electron distribution function is given by

$$f = f_0 \left\{ 1 - \frac{1}{v} \left[\left(\frac{mv^2}{2kT} \right) \left(\underline{v} \cdot \frac{\underline{\nabla} T}{T} \right) + \left(\frac{e}{kT} \right) (\underline{v} \cdot \underline{E}) \right] \right\}$$

Assume that $f = f_0 + f_1$, where $|f_1| \ll f_0$ and where f_0 is the following modified Maxwellian distribution

$$f_0(\underline{r}, v) = n(\underline{r}) \left[\frac{m}{2\pi kT(\underline{r})} \right]^{3/2} \exp \left[- \frac{mv^2}{2kT(\underline{r})} \right]$$

and neglect all second-order terms in the Boltzmann equation.

Consider the term involving $\underline{\nabla} f_1$ as a second order quantity.

11.11 - Using the distribution function of the previous problem, evaluate the electric current density $\underline{J}$ to show that the presence of a temperature gradient gives rise to an electric current associated with thermoelectric effects.

11.12 - Consider problem 11.10, but taking $\underline{E} = 0$ and, instead of the adiabatic case, consider the case of a constant kinetic pressure

$$p = n(\underline{r}) k T(\underline{r}) = \text{constant}$$

(a) Show that the electron distribution function is given by

$$f = f_0 \left\{ 1 - \frac{1}{v} \left[-\frac{5}{2} + \left(\frac{mv^2}{2kT} \right) \right] \underline{v} \cdot \frac{\underline{\nabla} T}{T} \right\}$$

(b) Evaluate the heat flux vector $\underline{q}$, and show that it can be written as

$$\underline{q} = -K \underline{\nabla} T(\underline{r})$$

where the thermal conductivity K is $5kp/2mv$.

(c) What is the value of the electric current density $\underline{j}$ in this case?

11.13 - In the previous problem, consider that $n = \text{constant}$ and that f_0 is the following modified Maxwell-Boltzmann distribution function

$$f_0(\underline{r}, v) = n \left[\frac{m}{2\pi kT(\underline{r})} \right]^{3/2} \exp \left[-\frac{mv^2}{2kT(\underline{r})} \right]$$

Calculate the electron distribution function $f(\underline{r}, \underline{v})$ and show that the heat flux vector is given by

$$\underline{q} = -K \underline{\nabla} T(\underline{r})$$

Determine the expression for the thermal conductivity K .

11.14 - In Problem 11.12, include the presence of an external magnetostatic field $\underline{B}$ in the z -direction and deduce the following expression for the nonequilibrium distribution function:

$$f = f_o \left\{ 1 - \frac{1}{v} \left(-\frac{5}{2} + \frac{mv^2}{2kT} \right) \left[\frac{v\omega_{ce}}{(v^2 + \omega_{ce}^2)} v_x \left(\frac{\hat{x}}{\omega_{ce}} - \hat{y} \right) + \frac{v\omega_{ce}}{(v^2 + \omega_{ce}^2)} v_y \left(\hat{x} + \frac{\hat{y}}{\omega_{ce}} \right) + v_z \hat{z} \right] \cdot \frac{\underline{\nabla} T}{T} \right\}$$

Show that the heat flux vector $\underline{q}$ can be expressed as

$$\underline{q} = -\underline{\underline{K}} \cdot \underline{\nabla} T(\underline{r})$$

where $\underline{\underline{K}}$ is the dyadic thermal conductivity, which in matrix form is given by

$$\underline{\underline{K}} = \begin{pmatrix} K_{\perp} & -K_H & 0 \\ K_H & K_{\perp} & 0 \\ 0 & 0 & K_{\parallel} \end{pmatrix}$$

with

$$K_{\perp} = \frac{v^2}{(v^2 + \omega_{ce}^2)} K_0$$

$$K_H = \frac{v\omega_{ce}}{(v^2 + \omega_{ce}^2)} K_0$$

$$K_{\parallel} = \frac{5 k p}{2 m v} \equiv K_0$$

Note: The solution of the differential equation

$$\frac{df_1}{d\phi} + \frac{v}{\omega_{ce}} f_1 = - \frac{1}{\omega_{ce}} \underline{v} \cdot \underline{\nabla} f_0$$

is given by

$$f_1 = - \exp\left(-\frac{v}{\omega_{ce}} \phi\right) \frac{1}{\omega_{ce}} \int_{-\infty}^{\phi} d\phi' \exp\left(\frac{v}{\omega_{ce}} \phi'\right) \underline{v} \cdot \underline{\nabla} f_0$$

11.15 - The *coefficient of viscosity* η is defined as the shear stress produced by unit velocity gradient. For the p_{xz} component of the kinetic pressure dyad we have

$$p_{xz} = - n \frac{d}{dz} u_x(z)$$

Assume the following form for the equilibrium velocity distribution function of the electrons

$$f_0(\underline{r}, \underline{v}) = n \left(\frac{m}{2\pi kT} \right)^{3/2} \exp \left[- \frac{m}{2kT} \left\{ [v_x - u_x(z)]^2 + v_y^2 + v_z^2 \right\} \right]$$

which indicates the presence of an average velocity $u_x(z)$ in the x-direction with a gradient in the z-direction. In the absence of external forces, and using the relaxation model for the collision term, with a constant collision frequency ν , let $f = f_0 + f_1$ with $|f_1| \ll f_0$ in the Boltzmann equation, under steady state conditions, to show that

$$f_1(\underline{r}, \underline{v}) = - \frac{2}{\nu} \left(\frac{m}{2kT} \right) f_0(\underline{r}, \underline{v}) v_z [v_x - u_x(z)] \frac{\partial}{\partial z} u_x(z)$$

Next, calculate p_{xz} and show that the coefficient of viscosity is given by

$$\eta = \frac{n k T}{\nu}$$