An extension of skeleton by influence zones and morphological interpolation to fuzzy sets

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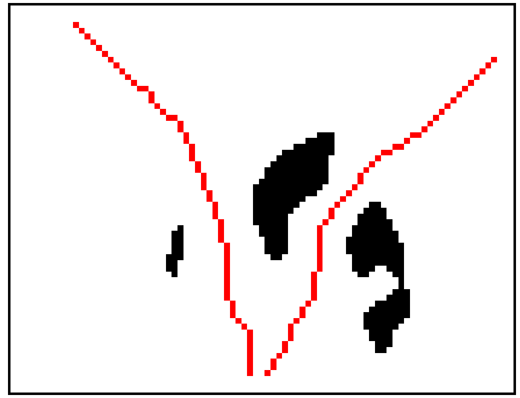
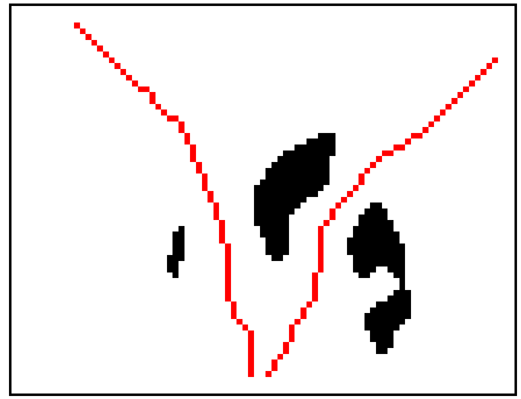
Motivation



- Notion of generalized Voronoï diagram or skeleton by influence zones (SKIZ)
- Extension to fuzzy sets
- Applications:
  - fuzzy influence zones
  - fuzzy notions of separation, space partitioning
  - fusion, interpolation
  - negotiations
  - "best" path
  - spatial reasoning on fuzzy regions of space
- Contributions:
  - definition of a fuzzy SKIZ
  - median fuzzy set and interpolation between fuzzy sets

SKIZ: binary (crisp) case



- $\mathcal{S}$: underlying space (spatial domain for instance)
- $X = \bigcup_i X_i$, $X_i \subseteq \mathcal{S}$, $X_i \cap X_j = \emptyset$ for $i \neq j$
- Influence zone of X_i:

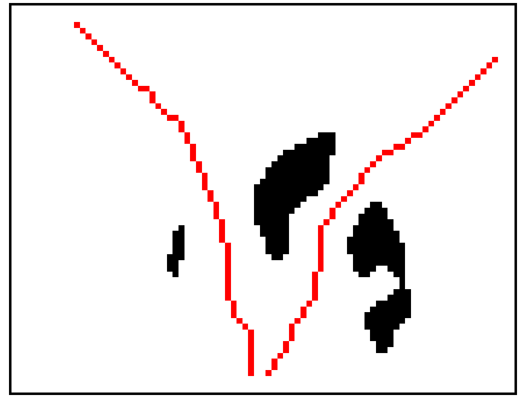
$$IZ(X_i) = \{x \in \mathcal{S} \; / \; d(x, X_i) < d(x, X \setminus X_i)\}$$

- Skeleton by influence zones:

$$\text{SKIZ}(X) = (\bigcup_i IZ(X_i))^c$$

= generalized Voronoï diagram

SKIZ: binary (crisp) case



- Expressions in terms of mathematical morphology:

$$IZ(X_i) = \bigcup_\lambda \left(\delta_\lambda(X_i) \cap \varepsilon_\lambda((\cup_{j \neq i} X_j)^c)\right) = \bigcup_\lambda \left(\delta_\lambda(X_i) \setminus \delta_\lambda(\cup_{j \neq i} X_j)\right)$$

$$\text{SKIZ}(X) = \text{WS}(d(y, X), y \in X^c)$$

How to extend these notions to the fuzzy case?

- Fuzzy sets on $\mathcal{S}$: μ_1, μ_2, ...
- Fuzzy mathematical morphology:
  - fuzzy structuring element ν (assumed to be symmetrical)
  - dilation and erosion:
    - dual definitions:
      $$\varepsilon_\nu(\mu^c) = (\delta_\nu(\mu))^c$$
    - example:
      $$\delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} \top(\mu(y), \nu(y-x))$$
      $\top$: t-norm (fuzzy intersection)
    - iterativity property:
      $$\delta_{2\nu}(\mu) = \delta_\nu(\delta_\nu(\mu)), \;\; \delta_{\lambda\nu}(\mu) = \delta_\nu(\delta_{(\lambda-1)\nu}(\mu))$$
    - $\nu(0) = 1 \Rightarrow \delta_\nu(\mu) \geq \mu$
- Distance to a fuzzy set:
  - fuzzy number $d(x, \mu)$
  - example: defined from a dilation

Fuzzy SKIZ from fuzzy dilations

Crisp case:

$$IZ(X_i) = \bigcup_\lambda \big(\delta_\lambda(X_i) \cap \varepsilon_\lambda((\cup_{j\neq i} X_j)^c)\big) = \bigcup_\lambda \big(\delta_\lambda(X_i) \setminus \delta_\lambda(\cup_{j\neq i} X_j)\big)$$

Extension to the fuzzy case for two sets:

$$IZ_{dil}(\mu_1) = \bigcup_\lambda (\delta_{\lambda\nu}(\mu_1) \cap \varepsilon_{\lambda\nu}(\mu_2^c)) = \bigcup_\lambda (\delta_{\lambda\nu}(\mu_1) \cap (\delta_{\lambda\nu}(\mu_2)^c) = \bigcup_\lambda (\delta_{\lambda\nu}(\mu_1) \setminus \delta_{\lambda\nu}(\mu_2))$$

Any number of fuzzy sets:

$$IZ_{dil}(\mu_i) = \bigcup_\lambda \big(\delta_{\lambda\nu}(\mu_i) \cap \varepsilon_{\lambda\nu}((\cup_{j\neq i}\mu_j)^c)\big)$$

Intersection: t-norm, union: t-conorm, complementation: $c(a) = 1 - a$

$$\Rightarrow \; IZ_{dil}(\mu_1) = \sup_\lambda \top[\delta_{\lambda\nu}(\mu_1), 1 - \delta_{\lambda\nu}(\mu_2)]$$

Fuzzy SKIZ (fuzzy generalized Voronoï diagram): $\boxed{\text{SKIZ}(\cup_i \mu_i) = (\bigcup_i IZ(\mu_i))^c}$

Fuzzy SKIZ from distances

Crisp case: $IZ(X_i) = \{x \in \mathcal{S} \ / \ d(x, X_i) < d(x, X \setminus X_i)\}$

Distance from point to a fuzzy set: $d(x,\mu)(n) = \top[\delta_{n\nu}(\mu)(x), 1 - \delta_{(n-1)\nu}(\mu)(x)]$ (discrete case) or $d(x,\mu)(\lambda) = \inf_{\lambda' < \lambda} \top[\delta_{\lambda\nu}(\mu)(x), 1 - \delta_{\lambda'\nu}(\mu)(x)]$ (continuous case)

Extension to the fuzzy case: comparison of fuzzy numbers

- Degree to which $d_1 < d_2$: $\mu(d_1 < d_2) = \sup_{a<b} \min(d_1(a), d_2(b))$
- Fuzzy influence zone of μ_1:

$$IZ_{dist1}(\mu_1)(x) = \mu(d(x,\mu_1) < d(x,\mu_2)) = \sup_{n<n'} \min[d(x,\mu_1)(n), d(x,\mu_2)(n')]$$

Extension to the fuzzy case: direct approach

- $d(x, X1) \leq d(x, X2)$: x is reached faster by dilating X_1 than by dilating X_2
- Degree to which $d(x,\mu_1)$ is less than $d(x,\mu_2)$:

$$\mu(d(x,\mu_1) \leq d(x,\mu_2)) = \inf_{\lambda} \perp[\delta_{\lambda\nu}(\mu_1)(x), 1 - \delta_{\lambda\nu}(\mu_2)(x)]$$

$$\mu(d(x,\mu_1) < d(x,\mu_2)) = 1 - \mu(d(x,\mu_2) \leq d(x,\mu_1))$$

- Fuzzy influence zone of μ_1:

$$IZ_{dist2}(\mu_1)(x) = 1 - \inf_{\lambda} \perp[\delta_{\lambda\nu}(\mu_2)(x), 1 - \delta_{\lambda\nu}(\mu_1)(x)]$$

Comparison of definitions and properties

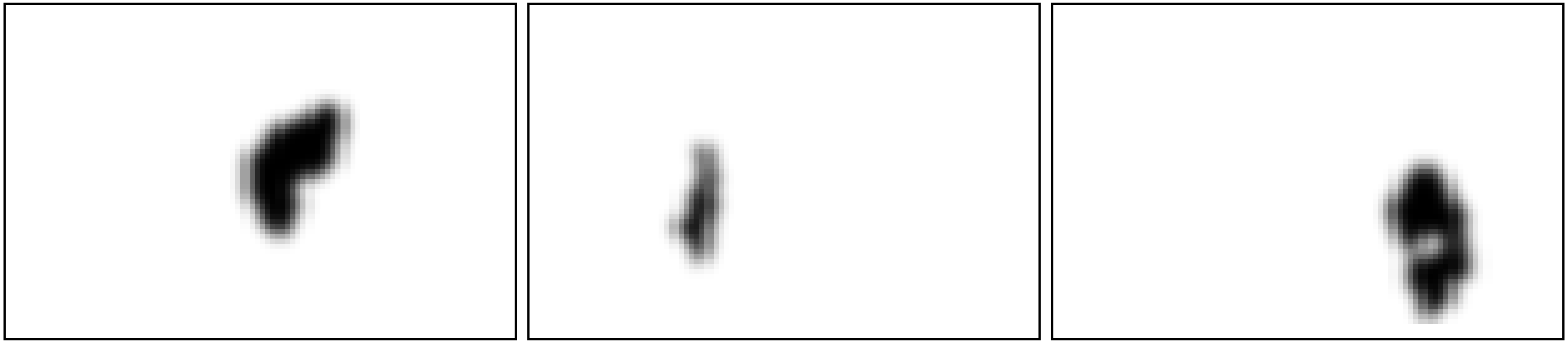
$$IZ_{dil}(\mu_1) = IZ_{dist2}(\mu_1)$$

$$IZ_{dist1}(\mu_1) \leq IZ_{dist2}(\mu_1)$$

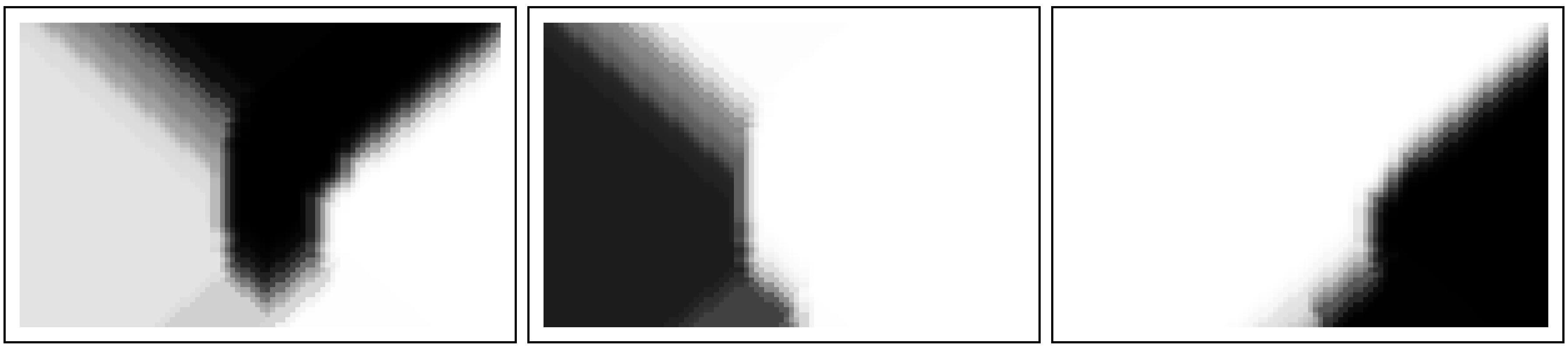
- Two interpretations: in terms of distance and of dilation.
- Complexity: the direct approach has a smaller computational cost.
- Equivalence with classical definitions when μ_i and ν are classical sets.
- SKIZ: independent of the order of μ_i.

Example

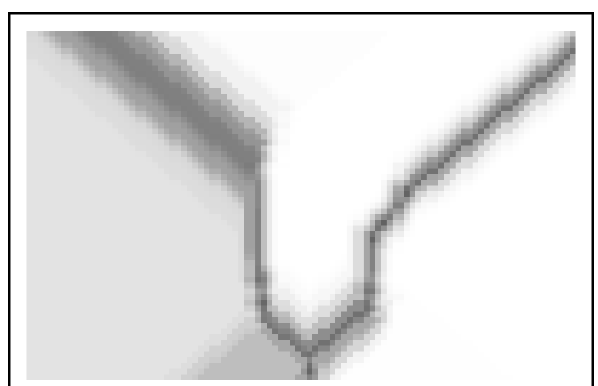
Three fuzzy objects:



Fuzzy influence zones (3×3 crisp structuring element):

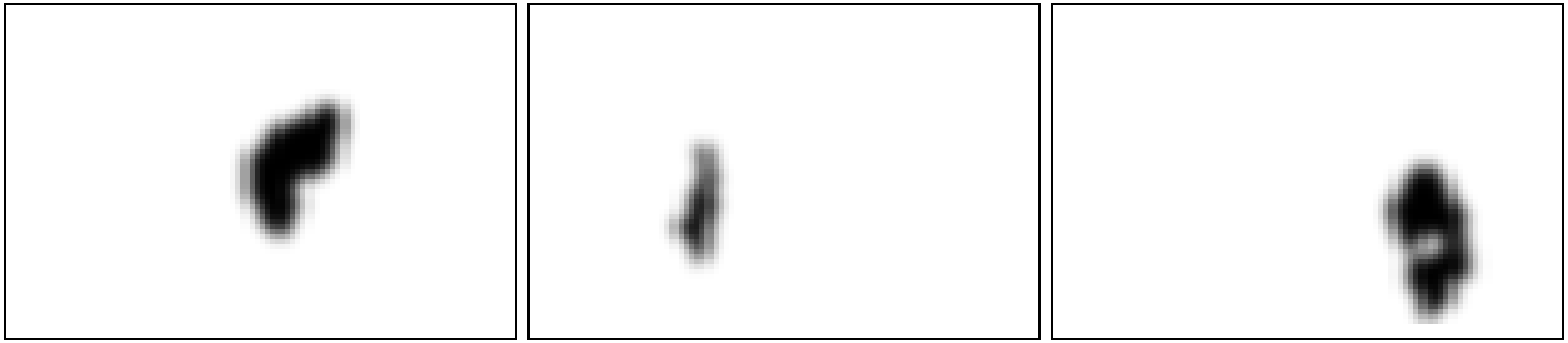


Fuzzy SKIZ

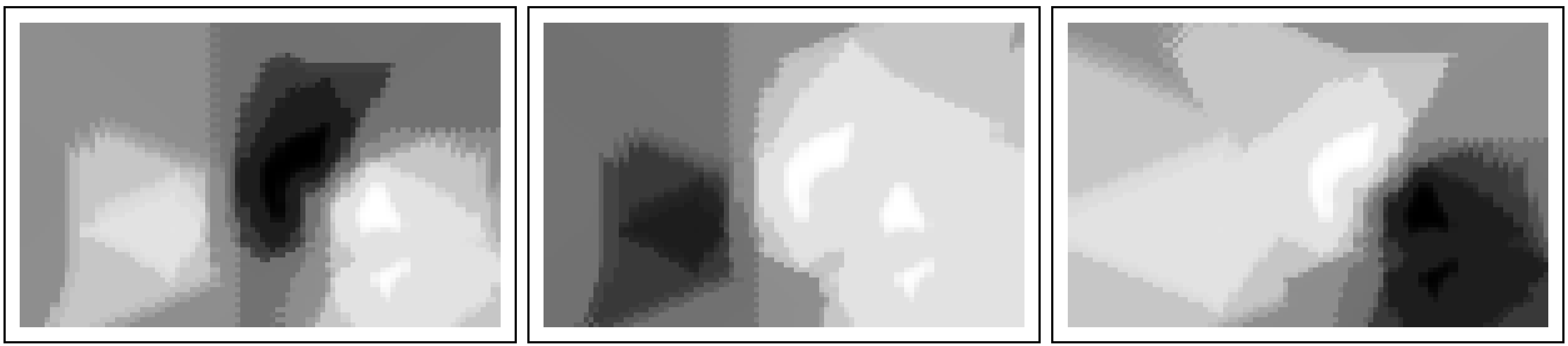


Example

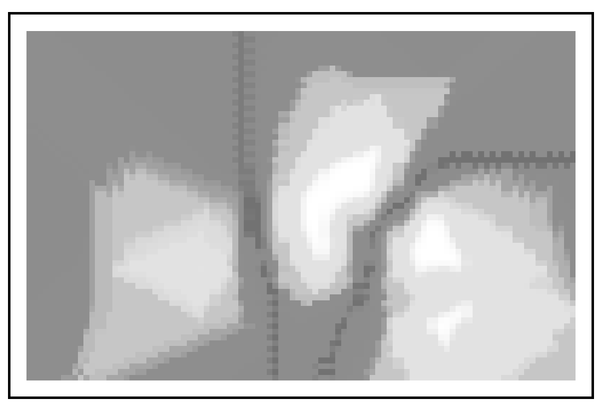
Three fuzzy objects:



Fuzzy influence zones (paraboloid fuzzy structuring element):

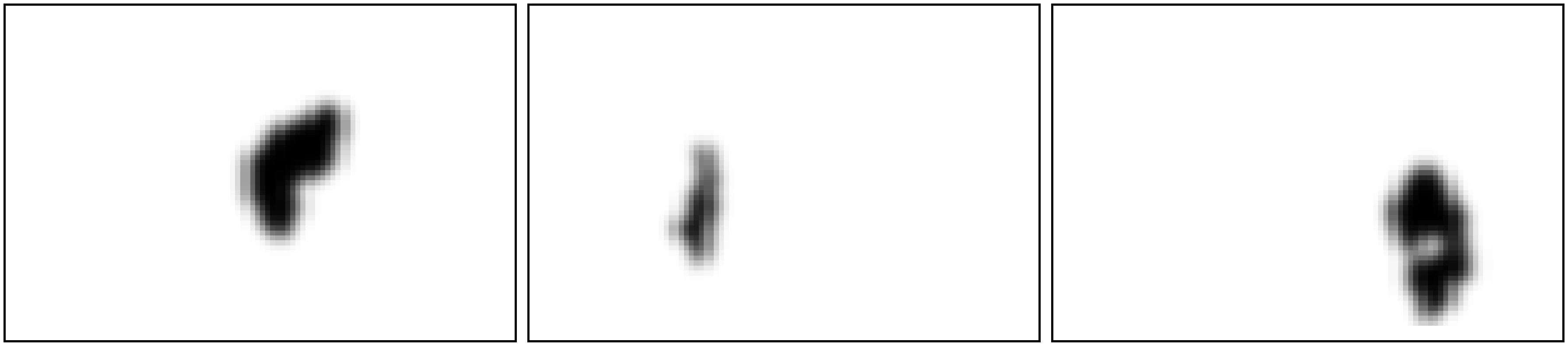


Fuzzy SKIZ:

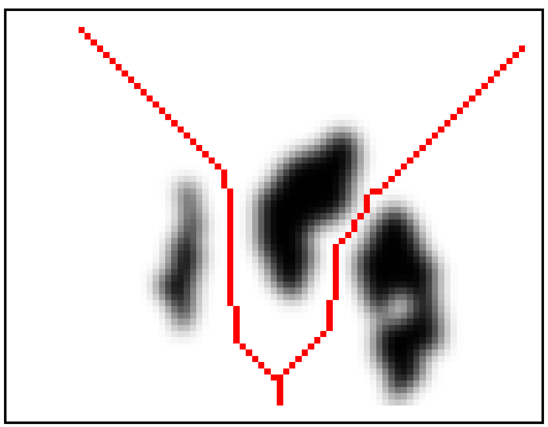


Example

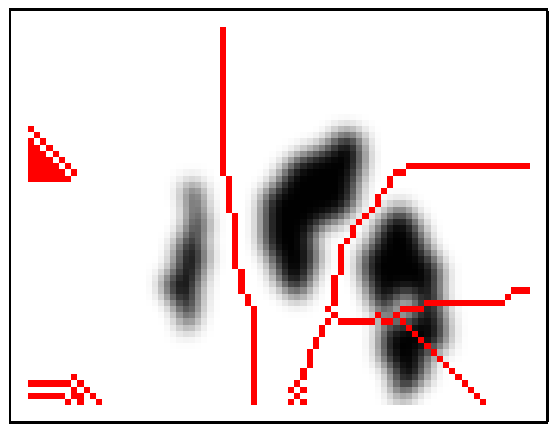
Three fuzzy objects:



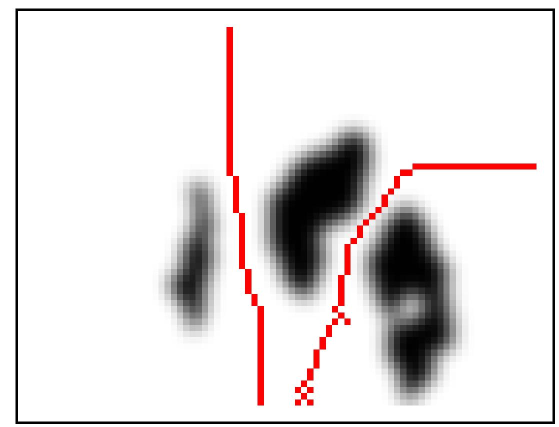
Binary decision using watersheds:



crisp structuring element

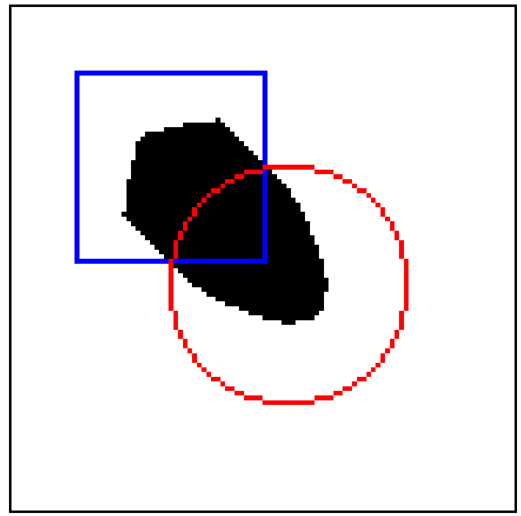


fuzzy structuring element



filtering of low degrees

Median set from the SKIZ



- $X \cap Y \neq \emptyset$
- $X_1 = X \cap Y$
- $X_2 = (X \cup Y)^c$
- SKIZ(X_1, X_2) and influence zone of X_1

Fuzzy median set and interpolation between fuzzy sets

- Influence zone of $\mu_1 \cap \mu_2$ with respect to $(\mu_1 \cup \mu_2)^c$
- Median from the dilatation-based definition:

$$M(\mu_1, \mu_2)(x) = \sup_\lambda \top[\delta_{\lambda\nu}(\mu_1 \cap \mu_2)(x), 1 - \delta_{\lambda\nu}((\mu_1 \cup \mu_2)^c)(x)]$$
$$= \sup_\lambda \top[\delta_{\lambda\nu}(\mu_1 \cap \mu_2)(x), \varepsilon_{\lambda\nu}(\mu_1 \cup \mu_2)(x)]$$

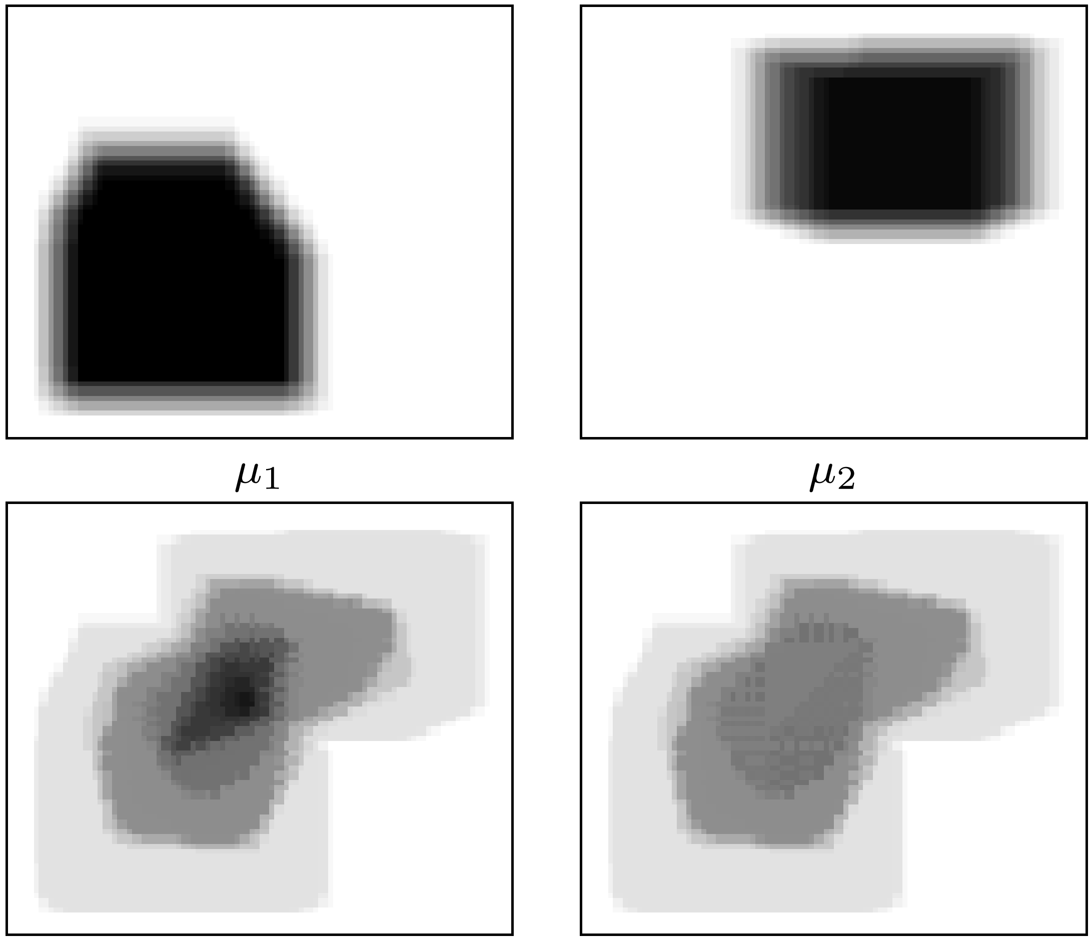
- Median from the definition based on the comparison of fuzzy numbers:

$$M'(\mu_1, \mu_2)(x) = \sup_{n<n'} \min[d(x, \mu_1 \cap \mu_2)(n), d(x, (\mu_1 \cup \mu_2)^c)(n')]$$

- Series of interpolating fuzzy sets between μ_1 and μ_2 :

$$\begin{aligned}
Interp(\mu_1, \mu_2)_{1/2} &= M(\mu_1, \mu_2) \\
Interp(\mu_1, \mu_2)_{1/4} &= M(Interp(\mu_1, \mu_2)_{1/2}, \mu_1) \\
Interp(\mu_1, \mu_2)_{1/8} &= M(Interp(\mu_1, \mu_2)_{1/4}, \mu_1) \\
&\dots \\
Interp(\mu_1, \mu_2)_{3/4} &= M(Interp(\mu_1, \mu_2)_{1/2}, \mu_2) \\
&\dots
\end{aligned}$$

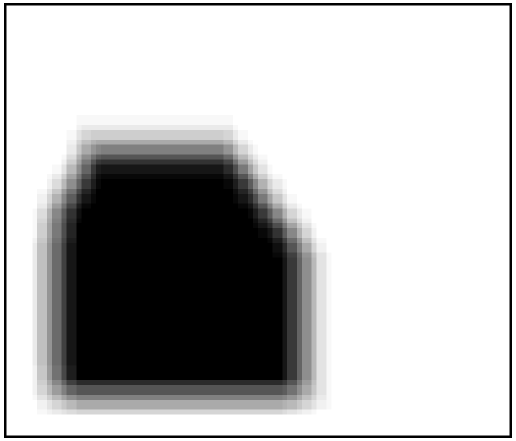
Example of fuzzy median set



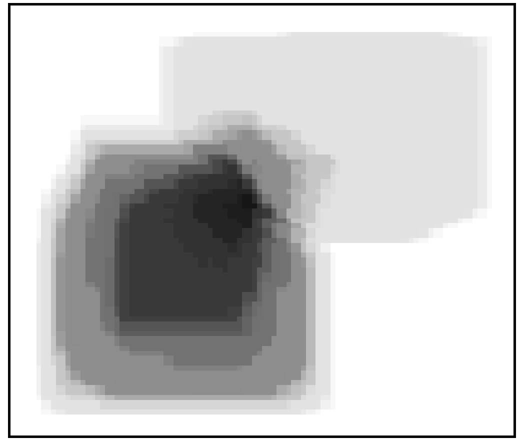
μ_1 μ_2

$M(\mu_1, \mu_2)$ $M'(\mu_1, \mu_2)$

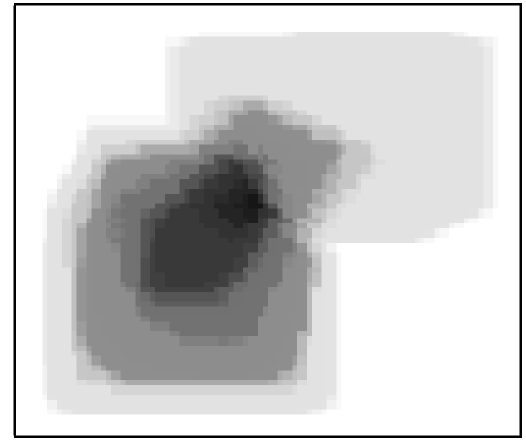
Example of interpolation between two fuzzy sets



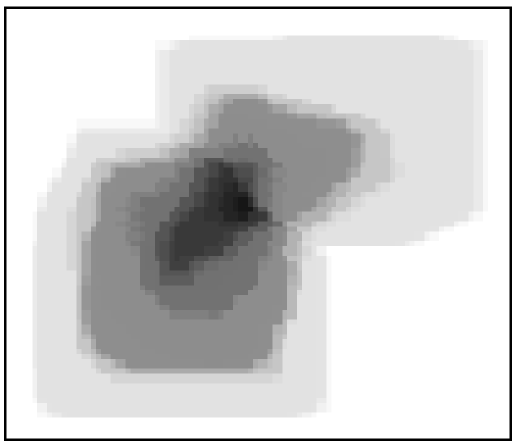
μ_1



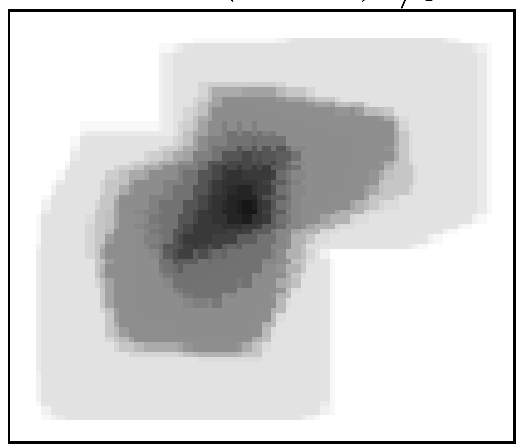
$Interp(\mu_1, \mu_2)_{1/8}$



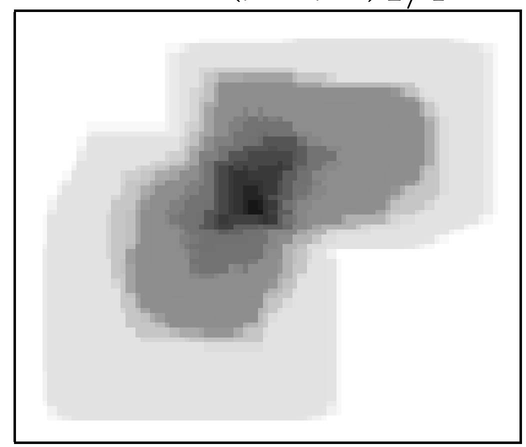
$Interp(\mu_1, \mu_2)_{1/4}$



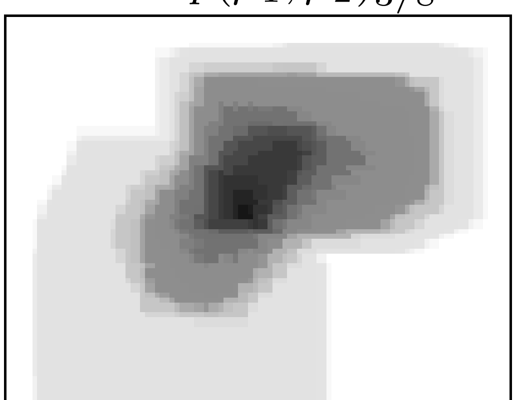
$Interp(\mu_1, \mu_2)_{3/8}$

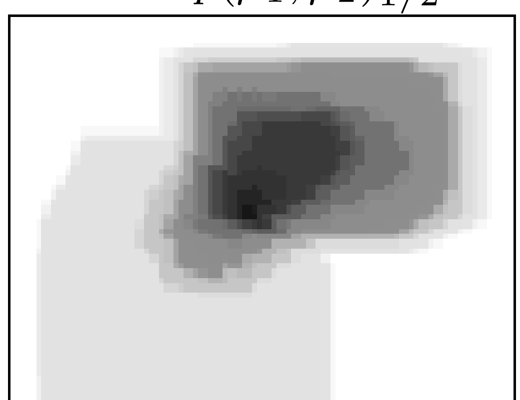


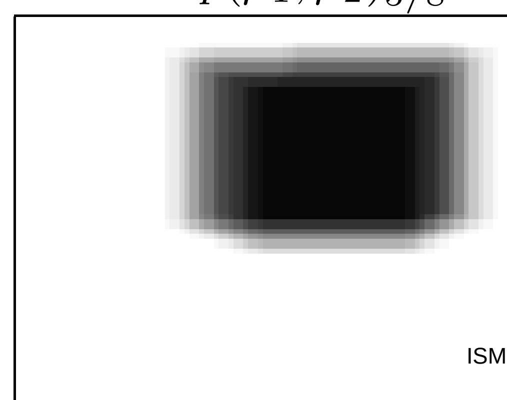
$Interp(\mu_1, \mu_2)_{1/2}$



$Interp(\mu_1, \mu_2)_{5/8}$







Some properties

- $\nu(0) = 1 \;\Rightarrow\; M(\mu_1, \mu_2) \leq (\mu_1 \cup \mu_2)$
- $\nu(0) = 1 \;\Rightarrow\; Core(\mu_1 \cap \mu_2) \subseteq Core(M(\mu_1, \mu_2)) \subseteq Core(\mu_1 \cup \mu_2)$
- $(\nu(x) = 1 \Leftrightarrow x = 0) \;\Rightarrow\; Supp(M(\mu_1, \mu_2)) = Supp(\mu_1 \cup \mu_2)$ et $Core(\mu_1 \cap \mu_2) = Core(M(\mu_1, \mu_2))$

Example: anatomical variability of brain structures

Example: merging preferences expressed in propositional logic

Models	abc	$\neg abc$	$a\neg bc$	$ab\neg c$	$\neg a\neg bc$	$\neg ab\neg c$	$a\neg b\neg c$	$\neg a\neg b\neg c$
φ_1	0	0	0	0	0	0.2	0	0
φ_2	0	0.5	0.5	0.5	0.5	0.8	0.5	0.7
$\delta_1(\varphi_1)$	0	0.2	0	0.2	0	0.2	0	0
$\varepsilon_1(\varphi_2)$	0	0	0	0	0.5	0.5	0.5	0.5
$\delta_2(\varphi_1)$	0.2	0.2	0	0.2	0.2	0.2	0.2	0.2
$\varepsilon_2(\varphi_2)$	0	0	0	0	0	0	0	0.5
$M(\varphi_1, \varphi_2)$	0	0	0	0	0	0.2	0	0.2

Conclusion

- Novel notions: fuzzy SKIZ, median fuzzy set and interpolation between fuzzy sets
- Morphological bases
- No hypothesis on $\mathcal{S}$ nor on the semantics of the fuzzy sets:
  - structuring element: distance or binary relation
  - multiple potential applications

- Extension to the case of non-intersecting fuzzy sets:
  - using translation in $\mathcal{S}$ (if this is meaningful)
  - otherwise: future work...
- Another approach: based on geodesic distances
- More on applications (compromise, negotiations, smoothing sets of preferences, etc.)

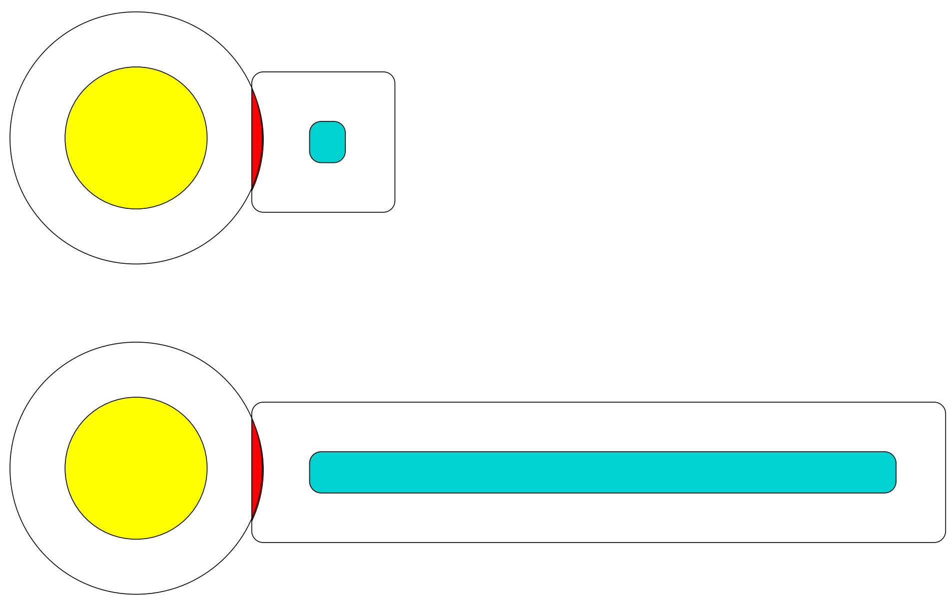
Example (ECAI 2006)

w_1:	1	1	1	1	1	1	1	1	1	1
w_2:	0	0	0	1	1	1	1	0	0	0
w_3:	0	0	0	0	0	0	0	1	1	1

$X = \{w_1, w_2\}$, $Y = \{w_1, w_3\}$ – Usual fusion $\Rightarrow \{w_1\}$

Why not:

w: 0 0 0 1 1 1 1 1 1 1 ?



$\Rightarrow$ Hausdorff distance and extension to the fuzzy case