Division of mappings between complete lattices

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This variety of sources for fundamental concepts has led to varying terminology and hence to difficulties in tracing history.

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Definition

A ***complete lattice*** is an ordered set such that any family $(x_j)_{j\in J}$ of elements possesses a smallest majorant and a largest minorant. We denote them by $\bigvee_{j\in J} x_j$ and $\bigwedge_{j\in J} x_j$, respectively.

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If Y is preordered, we define also its ***epigraph*** and its ***hypograph*** as

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It is often convenient to express properties of mappings in terms of their epigraphs or hypographs.

Definition

If two preordered sets X and Y and a mapping $f\colon X \to Y$ are given, we shall say that f is ***increasing*** if

$$\text{for all } x, x' \in X,\ x \leqslant_X x' \implies f(x) \leqslant_Y f(x'),$$

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To emphasize the symmetry between the two notions, we define, given any mapping $f\colon X \to Y$, a preorder $\leqslant_f$ in X by the requirement that $x \leqslant_f x'$ if and only if $f(x) \leqslant_Y f(x')$. Then f is increasing if and only if $\leqslant_X$ is finer than $\leqslant_f$, and f is coincreasing if and only if $\leqslant_f$ is finer than $\leqslant_X$.

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A mapping $f\colon\, L \longrightarrow M$ of a complete lattice L into a complete lattice M is said to be a ***dilation*** if $f\left(\bigvee_{j\in J} x_j\right) = \bigvee_{j\in J} f(x_j)$ for all families $(x_j)_{j\in J}$ of elements in L.

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Inverses of mappings

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Definition

Let L be a complete lattice, Y a preordered set, and $g\colon L \to Y$ any mapping. We then define the ***upper inverse*** $g^{[-1]}\colon Y \to L$ and the ***lower inverse*** $g_{[-1]}\colon Y \to L$ as the mappings

$$g^{[-1]}(y) = \bigwedge_{x \in L}(x; g(x) \geqslant_Y y) = \bigwedge_{x \in L}(x; (x,y) \in \operatorname{hypo} g), \quad y \in Y; \quad (1)$$

$$g_{[-1]}(y) = \bigvee_{x \in L}(x; g(x) \leqslant_Y y) = \bigvee_{x \in L}(x; (x,y) \in \operatorname{epi} g), \quad y \in Y. \quad (2)$$

We note that we always have

$$\left(\operatorname{epi} g^{[-1]}\right)^{-1} \supset \operatorname{hypo} g \quad \text{and} \quad \left(\operatorname{hypo} g_{[-1]}\right)^{-1} \supset \operatorname{epi} g. \tag{3}$$

Here $S^{-1} = \{(y, x) \in Y \times L; (x, y) \in S\}$ for any subset S of $L \times Y$.

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This special case has been studied for a long time, under several different names.

We shall see that there are many instances when we do not have this special case. After all, there are many mappings which are not dilations.

1. When the supremum in (2) is a maximum, the pair $(g, g_{[-1]})$ is said to be a *Galois connection* (Gierz & al. 2003:22), a concept which goes back to Évariste Galois' work on the automorphism groups of a field.

Singer (1997:172) calls a mapping $f\colon L \longrightarrow M$ a *duality* if $f\left(\bigwedge_{j\in J} x_j\right) = \bigvee_{j\in J} f(x_j)$ for all families $(x_j)_{j\in J}$ of elements in L. Thus a duality induces a dilation $L^{\text{op}} \to M$ and an erosion $L \to M^{\text{op}}$ if we change the order in L or M to the opposite order; the study of dualities in the sense of Singer is equivalent to that of dilations or erosions.

2. One also says in this special case when the supremum is a maximum that g is *residuated* and that $g_{[-1]}$ is its *residual* (Blyth & Janowitz 1972:11; Blyth 2005:7).

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2. One also says in this special case when the supremum is a maximum that g is *residuated* and that $g_{[-1]}$ is its *residual* (Blyth & Janowitz 1972:11; Blyth 2005:7). If the infimum in (1) is a minimum, g is said to be *dually residuated* and $g^{[-1]}$ is called its *dual residual*; the pair $(g^{[-1]}, g)$ is a Galois correspondence between Y and L. Residuation theory goes back at least to a paper by Ward & Dilworth (1939).

3. The pair $(g, g_{[-1]})$ is also said to be an *adjunction* (Gierz & al. 2003:22) in this special case. This aspect probably originates in logic, and is important in image processing.

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4. Finally, there is duality in convexity theory. The Fenchel transformation (Fenchel 1949) of a function $\varphi\colon \mathbb{R}^n \to [-\infty, +\infty]$ is defined as

$$\tilde{\varphi}(\xi) = \sup_{x \in \mathbb{R}^n} \big(\xi \cdot x - \varphi(x)\big), \qquad \xi \in \mathbb{R}^n,$$

and satisfies

$$\tilde{\varphi} \leqslant f \iff \tilde{f} \leqslant \varphi.$$

3. The pair $(g, g_{[-1]})$ is also said to be an *adjunction* (Gierz & al. 2003:22) in this special case. This aspect probably originates in logic, and is important in image processing.

4. Finally, there is duality in convexity theory. The Fenchel transformation (Fenchel 1949) of a function $\varphi\colon \mathbb{R}^n \to [-\infty, +\infty]$ is defined as

$$\tilde{\varphi}(\xi) = \sup_{x \in \mathbb{R}^n} \big(\xi \cdot x - \varphi(x)\big), \qquad \xi \in \mathbb{R}^n,$$

and satisfies

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After a change of order on one of the sides it satisfies (3) with equality, which means that we have a Galois correspondence.

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After a change of order on one of the sides it satisfies (3) with equality, which means that we have a Galois correspondence. It is also the case that

$$\big(\inf_{j \in J} \varphi\big)\tilde{} = \sup_{j \in J} \tilde{\varphi},$$

so that we have a duality in the sense of Singer; i.e., after a change of the order relation we have a dilation or erosion.

The results generalize residuation theory, equivalently the theory of Galois correspondences, to a more general situation, a situation which appears even in very simple examples as we shall see now.

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Example

Take $Y = L$ in the definition, fix an element c of L, and define a mapping $g\colon L \to L$ by $g(x) = x \vee c$, $x \in L$. In this case, the supremum in (2) is a maximum if $y \geqslant c$ but only then. Thus g is not residuated unless $c = \mathbf{0}$.

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Example

Take $Y = L$ in the definition, fix an element c of L, and define a mapping $g\colon L \to L$ by $g(x) = x \vee c$, $x \in L$. In this case, the supremum in (2) is a maximum if $y \geqslant c$ but only then. Thus g is not residuated unless $c = \mathbf{0}$. But it is easy to determine its lower inverse: $g_{[-1]}(y) = y$ if $y \geqslant c$ and $g_{[-1]}(y) = \mathbf{0}$ otherwise. We have

$$\operatorname{epi} g = \{(x, y) \in L^2; y \geqslant x \vee c\},$$

while

$$\left(\operatorname{hypo} g_{[-1]}\right)^{-1} = \operatorname{epi} g \cup \{(\mathbf{0}, y) \in L^2; y \not\geqslant c\},$$

so that

$$\left(\operatorname{hypo} g_{[-1]}\right)^{-1} \smallsetminus \operatorname{epi} g = \{(\mathbf{0}, y) \in L^2; y \not\geqslant c\} \neq \varnothing \text{ if } c \neq \mathbf{0}.$$

Example

Let now L be $\{0,1\}^2$ with the coordinatewise order, and let g be as before. We choose $c = (1,0)$ and denote $(0,1)$ by a so that L consists of the four element $\mathbf{0} = (0,0)$, $a = (0,1)$, $c = (1,0)$, and $\mathbf{1} = (1,1)$. We know that $g_{[-1]}(y) = y$ if $y \geqslant c$ and $g(y) = \mathbf{0}$ otherwise.

Example

Let now L be $\{0,1\}^2$ with the coordinatewise order, and let g be as before. We choose $c=(1,0)$ and denote $(0,1)$ by a so that L consists of the four element $\mathbf{0}=(0,0)$, $a=(0,1)$, $c=(1,0)$, and $\mathbf{1}=(1,1)$. We know that $g_{[-1]}(y)=y$ if $y \geqslant c$ and $g(y)=\mathbf{0}$ otherwise. We find that

$$\left(\operatorname{hypo} g_{[-1]}\right)^{-1} \smallsetminus \operatorname{epi} g=\{(\mathbf{0},\mathbf{0}),(\mathbf{0},a)\} \neq \varnothing.$$

Thus g is not residuated.

Properties of inverses

If, given a mapping $g\colon L \to Y$, we can find a mapping $u\colon Y \to L$ such that $\operatorname{epi} u = (\operatorname{hypo} g)^{-1}$ we would be content to have a kind of inverse to g. However, usually the best we can do is to study mappings with $\operatorname{epi} u \supset (\operatorname{hypo} g)^{-1}$ or $\operatorname{epi} v \subset (\operatorname{hypo} g)^{-1}$. This we shall do in the following proposition, which shows that the upper and lower inverses are solutions to certain extremal problems.

Proposition

Let L be a complete lattice, Y a preordered set, and let $g\colon L \to Y$, $u, v\colon Y \to L$ be mappings. If $\operatorname{epi} u \supset (\operatorname{hypo} g)^{-1} \supset \operatorname{epi} v$, then $u \leqslant g^{[-1]} \leqslant v$ and

$$\operatorname{epi} u \supset \operatorname{epi} g^{[-1]} \supset (\operatorname{hypo} g)^{-1} \supset \operatorname{epi} v.$$

Hence $g^{[-1]}$ is the largest mapping u such that $\operatorname{epi} u$ contains $(\operatorname{hypo} g)^{-1}$.

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Hence $g^{[-1]}$ is the largest mapping u such that $\operatorname{epi} u$ contains $(\operatorname{hypo} g)^{-1}$. Similarly, if $\operatorname{hypo} u \subset (\operatorname{epi} g)^{-1} \subset \operatorname{hypo} v$, then $u \leqslant g_{[-1]} \leqslant v$ and

$$\operatorname{hypo} u \subset (\operatorname{epi} g)^{-1} \subset \operatorname{hypo} g_{[-1]} \subset \operatorname{hypo} v.$$

Hence $g_{[-1]}$ is the smallest mapping v which satisfies $\operatorname{hypo} v \supset (\operatorname{epi} g)^{-1}$.

Corollary

With g, u and v given as in the proposition, assume that $(\operatorname{epi} u)^{-1} = \operatorname{hypo} g$. Then $u = g^{[-1]}$. Similarly, if $(\operatorname{hypo} v)^{-1} = \operatorname{epi} g$, then $v = g_{[-1]}$.

Corollary

With g, u and v given as in the proposition, assume that $(\operatorname{epi} u)^{-1} = \operatorname{hypo} g$. Then $u = g^{[-1]}$. Similarly, if $(\operatorname{hypo} v)^{-1} = \operatorname{epi} g$, then $v = g_{[-1]}$. If also Y is a complete lattice, then $\operatorname{epi} u = (\operatorname{hypo} g)^{-1}$ implies that $u_{[-1]} = g$ in addition to $u = g^{[-1]}$.

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The corollary singles out the special case of adjunctions between L and Y among all pairs $\left(g, g_{[-1]}\right)$ and adjunctions between Y and L among all pairs $\left(g^{[-1]}, g\right)$.

An ideal inverse u would satisfy $u \circ g = \mathrm{Id}_L$, $g \circ u = \mathrm{Id}_Y$, and the inverse of u would be g. It is therefore natural to compare $g^{[-1]} \circ g$ and $g_{[-1]} \circ g$ with Id_L; $g \circ g^{[-1]}$ and $g \circ g_{[-1]}$ with Id_Y; and $\left(g_{[-1]}\right)^{[-1]}$ and $\left(g^{[-1]}\right)_{[-1]}$ with g. This is what we shall do next.

Left inverses

We shall now investigate to what extent $g^{[-1]}$ and $g_{[-1]}$ can serve as left inverses to g.

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Theorem

Let L be a complete lattice and Y a preordered set. Then the following six conditions are equivalent.
(a) *g is coincreasing;*
(b) $g^{[-1]} \circ g \geqslant \mathrm{Id}_L$;
(c) $g^{[-1]} \circ g = \mathrm{Id}_L$;
(d) $g_{[-1]} \circ g \leqslant \mathrm{Id}_L$;
(e) $g_{[-1]} \circ g = \mathrm{Id}_L$;
(f) $g_{[-1]} \leqslant g^{[-1]}$.

Right inverses

Next we compose $g_{[-1]}$ with g in the other order: we shall see to what extent the inverses we have constructed can serve as right inverses. This will lead to a characterization of dilations—and, by duality, of erosions.

Right inverses

Next we compose $g_{[-1]}$ with g in the other order: we shall see to what extent the inverses we have constructed can serve as right inverses. This will lead to a characterization of dilations—and, by duality, of erosions.

Theorem

If L and M are complete lattices and $g\colon L \longrightarrow M$ is any mapping, then the following five properties are equivalent.
(A) *g is a dilation;*
(B) $\big(\operatorname{hypo}(g_{[-1]})\big)^{-1} \subset \operatorname{epi} g$*;*
(C) $\big(\operatorname{hypo}(g_{[-1]})\big)^{-1} = \operatorname{epi} g$*;*
(D) *g is increasing and* $\big(\operatorname{graph}(g_{[-1]})\big)^{-1} \subset \operatorname{epi} g$*;*
(E) *g is increasing and $g \circ g_{[-1]} \leqslant \mathrm{Id}_M$.*

This theorem characterizes the special case when the supremum in (2) is a maximum (property (E)); equivalently, it characterizes the special case of residuated mappings or Galois correspondence (property (C)).

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By duality we get a similar characterization of erosions; equivalently of the case when the infimum defining the upper inverse is a minimum.

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By duality we get a similar characterization of erosions; equivalently of the case when the infimum defining the upper inverse is a minimum.

Corollary

If L and M are complete lattices and $g\colon L \to M$ and $u\colon M \to L$ are two mappings such that $\operatorname{epi} g = (\operatorname{hypo} u)^{-1}$, then u is a dilation and g is an erosion, and $g_{[-1]} = u$, $u^{[-1]} = g$.

Inverses of inverses

Theorem

If L and M are complete lattices and $g\colon L \to M$ is any mapping, then quite generally $\left(g_{[-1]}\right)^{[-1]} \leqslant g \leqslant \left(g^{[-1]}\right)_{[-1]}$. Equality holds at the first place if and only if g is a dilation; at the second place if and only if g is an erosion.

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Theorem

If L and M are complete lattices and $\delta\colon L \to M$ is a dilation, then $\delta_{[-1]}\colon M \to L$ is an erosion. Similarly, if $\varepsilon\colon L \to M$ is an erosion, then $\varepsilon^{[-1]}$ is a dilation.

A well-known consequence:

Corollary

For any dilation $\delta\colon L \to M$ *we have* $\delta \circ \delta_{[-1]} \circ \delta = \delta$ *and* $\delta_{[-1]} \circ \delta \circ \delta_{[-1]} = \delta_{[-1]}$. *In particular,* $\delta_{[-1]} \circ \delta$ *and* $\delta \circ \delta_{[-1]}$ *are idempotent and therefore ethmomorphisms. The first is a cleistomorphism in* L*, the second an anoiktomorphism in* M.

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Division of mappings

We shall now generalize the definitions of upper and lower inverses.

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Definition

Let a set X, a complete lattice M, and a preordered set Y, as well as two mappings $f\colon X \to M$ and $g\colon X \to Y$ be given. We define two mappings $f /^{\star} g, f /_{\star} g\colon Y \to M$ by

$$(f /^{\star} g)(y) = \bigwedge_{x \in X} (f(x); g(x) \geqslant_Y y), \qquad y \in Y,$$

$$(f /_{\star} g)(y) = \bigvee_{x \in X} (f(x); g(x) \leqslant_Y y), \qquad y \in Y.$$

We shall call them the ***upper quotient*** and the ***lower quotient*** of f and g.

The quotients $f \,/^{\star} g$ and $f \,/_{\star}\, g$ increase when f increases and they decrease when g increases—just as with division of positive numbers:

If $f_1 \leqslant_M f_2$ and $g_1 \geqslant_Y g_2$, then $f_1 \,/^{\star} g_1 \leqslant_M f_2 \,/^{\star} g_2$ and $f_1 \,/_{\star}\, g_1 \leqslant_M f_2 \,/_{\star}\, g_2$.

If we specialize the definitions to the situation when $X = M$ and $f = \mathrm{Id}_X$, then $f \,/^{\star} g = \mathrm{Id}_X \,/^{\star} g = g^{[-1]}$ and $f \,/_{\star}\, g = \mathrm{Id}_X \,/_{\star}\, g = g_{[-1]}$.

Proposition

Let X be an arbitrary subset of a complete lattice M, let $Y = M$, and g the inclusion mapping $X \to M$. Then $f \,/^{\star} g = f^{\diamond}$ and $f \,/_{\star}\, g = f_{\diamond}$, where $f^{\diamond}$ is the largest increasing mapping $h \colon M \to M$ such that $h\big|_X$ minorizes f, i.e.,

$$f^{\diamond}(y) = \sup_h \big(h(y); h \text{ is increasing and } h(x) \leqslant f(x) \text{ for all } x \in X\big);$$

and $f_{\diamond}$ is the smallest increasing mapping k such that $k\big|_X$ majorizes f, i.e.,

$$f_{\diamond}(y) = \inf_k \big(k(y); k \text{ is increasing and } k(x) \geqslant f(x) \text{ for all } x \in X\big).$$

If f itself is increasing, they are in fact extensions of f.

If we specialize further, letting also f be the inclusion mapping $X \to M$, we obtain

$$(f \not\star g)(y) = (f \not\star f)(y) = f_\diamond(y) = \bigvee_{x \in X} (x; x \leqslant y) = y^\circ \in M,$$

where the last equality defines y°. It is easy to verify that $y \mapsto y^\circ$ is an anoiktomorphism. A well-known situation is described in the following example.

If we specialize further, letting also f be the inclusion mapping $X \to M$, we obtain

$$(f /_{\star} g)(y) = (f /_{\star} f)(y) = f_{\diamond}(y) = \bigvee_{x \in X} (x; x \leqslant y) = y^{\circ} \in M,$$

where the last equality defines y°. It is easy to verify that $y \mapsto y^{\circ}$ is an anoiktomorphism. A well-known situation is described in the following example.

Example

Let M be the complete lattice $[-\infty, +\infty]^E$ of functions on a vector space E with values in the extended reals, let F be a vector subspace of its algebraic dual $E^{\star}$ (the space of all linear forms on E), and let X be the set of all affine functions with linear part in F, i.e., functions of the form $\alpha(x) = \xi(x) + c$ for some linear form $\xi \in F$ and some real constant c. A function f such that $f^{\circ} = f$ is called X*-convex* by Singer (1997:10).

We next compare the quotient $f /^{\star} g$ and the composition $f \circ g^{[-1]}$ (think of $x/y = x \cdot y^{-1}$ for positive numbers):

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Proposition

For every increasing mapping $f\colon X \to M$ and every mapping $g\colon X \to Y$ we have $f /^{\star} g \geqslant f \circ g^{[-1]}$ with equality if f is an erosion, and $f /_{\star} g \leqslant f \circ g_{[-1]}$ with equality if f is a dilation. If g is coincreasing, then $f /_{\star} g \leqslant f \circ g_{[-1]} \leqslant f \circ g^{[-1]} \leqslant f /^{\star} g$.

Proposition

If P is a preordered set and $h\colon M \to P$ is increasing, we have $h \circ (f /^{\star} g) \leqslant (h \circ f) /^{\star} g$ with equality if h is an erosion.

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Pullbacks and pushforwards

We shall now see how the notions introduced fit into the study of a very fundamental situation, that of a mapping $f\colon X \to Y$ pulling back mappings $\psi\colon Y \to L$ and pushing forward mappings $\varphi\colon X \to L$.

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Definition

Let three sets X, Y and L be given, as well as a mapping $f\colon X \to Y$. We define the ***pullback*** of f, denoted by $f^{\leftarrow}\colon L^Y \to L^X$ and having as values mappings $f^{\leftarrow}(\psi)\colon X \to L$, by

$$f^{\leftarrow}(\psi) = \psi \circ f \in L^X, \qquad \psi \in L^Y. \tag{4}$$

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$$f^{\leftarrow}(\psi) = \psi \circ f \in L^X, \qquad \psi \in L^Y. \tag{4}$$

One often writes f^* for $f^{\leftarrow}$.

Definition

Let two sets X, Y and a complete lattice L be given, as well as a mapping $f\colon X \to Y$. We define the ***upper*** and ***lower pushforwards*** denoted by $f^{\to}, f_{\to}\colon L^X \to L^Y$ and yielding as values mappings $f^{\to}(\varphi), f_{\to}(\varphi)\colon Y \to L$, by

$$f^{\to}(\varphi)(y) = \bigwedge_{x\in X} \big(\varphi(x); f(x) = y\big), \qquad y \in Y, \quad \varphi \in L^X, \tag{5}$$

and

$$f_{\to}(\varphi)(y) = \bigvee_{x\in X} \big(\varphi(x); f(x) = y\big), \qquad y \in Y, \quad \varphi \in L^X. \tag{6}$$

If $y \notin \mathrm{im} f$, we obtain $f^{\rightarrow}(\varphi)(y) = \mathbf{1}_L$; the infimum is not attained. Similarly, $f_{\rightarrow}(\varphi)(y) = \mathbf{0}_L$. Of course there are many other cases when the supremum and infimum in (5), (6) are not attained.

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We note that the upper pushforward defined by (5) is actually an upper quotient, $f^{\rightarrow}(\varphi) = \varphi /^{\star} f$, and similarly $f_{\rightarrow}(\varphi) = f /_{\star} f$, namely if we provide Y with the discrete order. The results on quotients are therefore available in this setting.

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Proposition

Let $f \colon X \to Y$ and a complete lattice L be given. Then the pullback $f^{\leftarrow} \colon L^Y \to L^X$ is both a dilation and an erosion.

Proposition

If X, Y are sets, $f\colon X \to Y$ a mapping, and L a complete lattice, then

$$\operatorname{epi} f^{\leftarrow} = \left(\operatorname{hypo} f^{\rightarrow}\right)^{-1} \tag{7}$$

and

$$\operatorname{hypo} f^{\leftarrow} = \left(\operatorname{epi} f_{\rightarrow}\right)^{-1}. \tag{8}$$

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Corollary

Let a mapping $f\colon X \to Y$ and a complete lattice L be given. Then the lower inverse of the pullback is $(f^{\leftarrow})_{[-1]} = f^{\rightarrow}$, and the supremum in the definition of the lower inverse is attained. Also the upper inverse of $f^{\rightarrow}$ is $(f^{\rightarrow})^{[-1]} = f^{\leftarrow}$.

Corollary

Let a mapping $f\colon X \longrightarrow Y$ and a complete lattice L be given. Then the lower inverse of the pullback is $(f^{\leftarrow})_{[-1]} = f^{\rightarrow}$, and the supremum in the definition of the lower inverse is attained. Also the upper inverse of $f^{\rightarrow}$ is $(f^{\rightarrow})^{[-1]} = f^{\leftarrow}$.

Corollary

Let a mapping $f\colon X \longrightarrow Y$ and a complete lattice L be given. Then the upper inverse of the pullback is $(f^{\leftarrow})^{[-1]} = f_{\rightarrow}$, and the infimum in the definition of the upper inverse is attained. Also $(f_{\rightarrow})_{[-1]} = f^{\leftarrow}$.

Corollary

Let a mapping $f\colon X \to Y$ and a complete lattice L be given. Then the lower inverse of the pullback is $(f^{\leftarrow})_{[-1]} = f^{\rightarrow}$, and the supremum in the definition of the lower inverse is attained. Also the upper inverse of $f^{\rightarrow}$ is $(f^{\rightarrow})^{[-1]} = f^{\leftarrow}$.

Corollary

Let a mapping $f\colon X \to Y$ and a complete lattice L be given. Then the upper inverse of the pullback is $(f^{\leftarrow})^{[-1]} = f_{\rightarrow}$, and the infimum in the definition of the upper inverse is attained. Also $(f_{\rightarrow})_{[-1]} = f^{\leftarrow}$.

Corollary

The upper pushforward mapping $f^{\rightarrow}$ is a dilation, and the lower pushforward $f_{\rightarrow}$ is an erosion.

We consider now the special case when $L = \{0, 1\}$ and ψ is a characteristic function, $\psi = \chi_B \in \{0, 1\}^Y$ for some subset B of Y. Then

$$f^{\leftarrow}(\chi_B) = \chi_A, \text{ where } A = \{x \in X; f(x) \in B\},$$

the preimage of B.

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If φ is a characteristic function $\varphi = \chi_A$ for a subset of X, then

$$f_{\rightarrow}(\chi_A) = \chi_B, \text{ where } B = \{f(x); x \in A\},$$

the direct image of A under f, and

$$f^{\rightarrow}(\chi_A) = \chi_C, \text{ where } C = Y \smallsetminus \{f(x); x \notin A\},$$

the complement of the direct image of the complement of A.

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the direct image of A under f, and

$$f^{\rightarrow}(\chi_A) = \chi_C, \text{ where } C = Y \smallsetminus \{f(x); x \notin A\},$$

the complement of the direct image of the complement of A. Because of this it is natural to define also a pullback $f^{\leftarrow}\colon \mathscr{P}(Y) \to \mathscr{P}(X)$ and pushforwards $f^{\rightarrow}, f_{\rightarrow}\colon \mathscr{P}(X) \to \mathscr{P}(Y)$. There is in general no inclusion relation between the two sets B and C defined by $f_{\rightarrow}(\chi_A)$ and $f^{\rightarrow}(\chi_A)$, thus between $f_{\rightarrow}(A)$ and $f^{\rightarrow}(A) = \complement f_{\rightarrow}(\complement A)$.

Cleistomorphisms and anoiktomorphisms

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Theorem

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Thank you for your interest!