Locally finite spaces and the join operator

Erik Melin

Uppsala University, Department of Mathematics

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- We show that such a space can be *uniquely* decomposed as a *join* of indecomposable spaces.
- A long-term goal is to understand global structure. What are the possibilities to introduce digital manifolds?

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- Digital images are (sometimes) topological spaces. We might deal with complicated spaces.
- We would (perhaps) like morphological operations compatible with the structure of the spaces, for example connectivity or closure operations. The formalism and the theorems proved about this formalism could be a useful tool.

Main results

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Theorem (Uniqueness)

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A locally finite space can be written in a unique way as $Y_1 \vee Y_2 \vee \cdots \vee Y_n$, where each Y_i is indecomposable.

Theorem (Cancellation)

If $A \vee X$ and $B \vee X$ are homeomorphic, then A and B are homeomorphic.

Axioms for a smallest-neighborhood space

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The spaces are called **smallest-neighborhood spaces** or **Alexandrov spaces**.

Smallest-neighborhoods

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- If X a smallest-neighborhood space, then $\mathscr{N}(x)$ is open.
- $\mathscr{C}(x)$ is always closed.
- $\{x, y\}$ *connected* if and only if $x \in \mathrm{AN}(y)$.

Locally finite spaces

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A space X is countable if and only if it is locally countable and has countably many connectivity components.

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If X is locally finite, then the set of open points in X is dense in X.

If a space is not locally finite, then this need not be the case.

Example

Consider $\mathbb{Z}$ equipped with the open sets $]-\infty, m]$, $m \in \mathbb{Z}$. This space has no open point.

Proof of countability

A **Khalimsky arc** in X is the homeomorphic image of a Khalimsky interval, $[a, b]_{\mathbb{Z}}$. The **length** of an arc is the number of points minus 1. The arc metric on X (provided X is T_0 and connected) is the following:

$$\rho(x, y) = \min(\mathrm{Length}(A);\ A \text{ is a Kh. arc containing } x \text{ and } y.)$$

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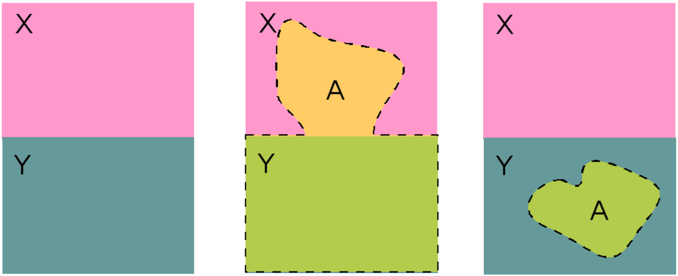
Every ball is finite, and $X = \bigcup_{m \in \mathbb{Z}} B_m$.

The join of two spaces

Let X and Y be topological spaces.

The join, $X \vee Y$, is the disjoint (set) union where $A \subset X \dot{\cup} Y$ is declared to be open if:

- $A \cap X$ open in X, $A \cap Y = Y$, or
- $A \cap Y$ open in Y, $A \cap X = \varnothing$.



The join in smallest-neighborhood spaces

Suppose $Z = X \vee Y$ where X and Y are smallest-neighborhood spaces. Then

- $\mathscr{N}_Z(x) = \mathscr{N}_X(x) \cup Y$ if $x \in X$.
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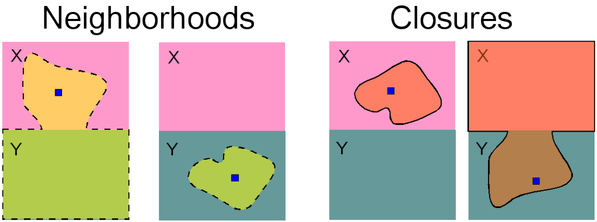
Dually, the closures become.

- $\mathscr{C}_Z(x) = \mathscr{C}_Y(x)$ if $x \in X$.
- $\mathscr{C}_Z(y) = \mathscr{C}_X(y) \cup X$ if $y \in Y$.



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- $\forall X, Y, Z \colon (X \vee Y) \vee Z = X \vee (Y \vee Z)$ (associative).
- $X \vee Y$ is always connected, provided $X \neq \varnothing$ and $Y \neq \varnothing$.

Decomposable spaces

Definition

Z is called **indecomposable** if $Z = X \vee Y$ implies $X = \varnothing$ or $Y = \varnothing$. Otherwise Z is called **decomposable**.

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The Khalimsky interval $I = [1,3]_{\mathbb{Z}}$ is decomposable. $I = \{2\} \vee \{1,3\}$ where the factors have the discrete topology.

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Theorem (Uniqueness)

Suppose that Z is locally finite and T_0. If $Z = X \vee Y$, $Z = \tilde{X} \vee \tilde{Y}$ and if X, $\tilde{X}$ are indecomposable, then $X = \tilde{X}$ and $Y = \tilde{Y}$.

Decomposable spaces continued

Can we somehow recognize indecomposable spaces?

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Theorem

Let Z be a smallest-neighborhood space. If Z is decomposable, then $\rho(x,y) \leqslant 2$ for all $x, y \in Z$.

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Example

The Khalimsky interval $I = [1, 4]_{\mathbb{Z}}$ is indecomposable.

Conclusion: The join is only of interest with a local perspective.

Cancellation laws

It is not true, in general, that we from $A \vee X \approx B \vee X$ can conclude that $A \approx B$. However, if X is locally finite, we have

Theorem

Let X be locally finite. Then for every smallest-neighborhood space A and B we have

$$A \vee X \approx B \vee X \implies A \approx B$$

and

$$X \vee A \approx X \vee B \implies A \approx B$$

Some observations

Claim

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Given $x \in \mathbb{Z}^n$ (Khalimsky n-space), we have $\mathscr{N}(x) \approx [-1, +1]^k$ and $\mathscr{C}(x) \approx [0, 2]^l$.

Understanding $\mathscr{A}(x)$ directly is relatively complicated...

(The characterization of $\mathbb{Z}^n$ can be found in Evako et al. (1996))
Note: Unique decomposition immediately gives uniqueness of dimension.

Partial orders

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Viewed as partial orders $X \vee Y$ means X is placed on top of Y.

$\mathscr{A}(x) = (\mathscr{C}(x) \cup \mathscr{N}(x)) \smallsetminus \{x\} = = \{y;\ y \succ x\} \cup \{y;\ y \prec x\}$

The first claim follows. The second claim is similar.

Consider again the second claim:
$\mathrm{AN}(x) = (\mathscr{C}(x) \smallsetminus \{x\}) \vee \{x\} \vee (\mathscr{N}(x) \smallsetminus \{x\})$.

The join and partial orders

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From this follows: The set of open points in X is dense in X.

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- For many results we have to assume local finiteness, that is, finite neighborhoods.
- Next project: Understand the possibilities (or impossibility) of creating a (reasonable) digital Khalimsky manifold concept.