Watershed cuts

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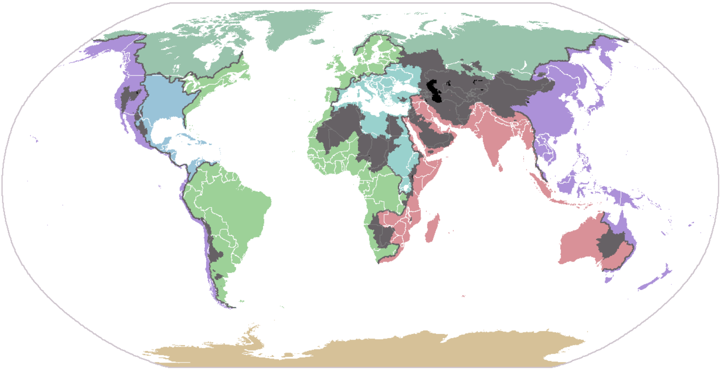
ISMM 2007
October 13th

Context



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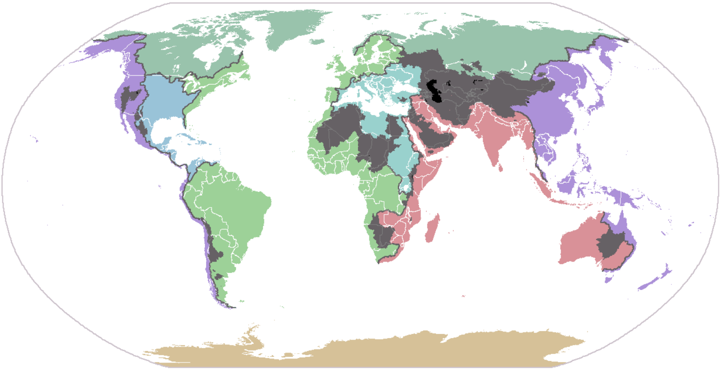

- For topographic purposes, the watershed has been studied since the 19th century (Maxwell, Jordan, . . .)

Context

- One hundred years later (1978), it was introduced by Digabel and Lantuéjoul for image segmentation

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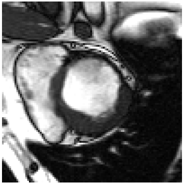

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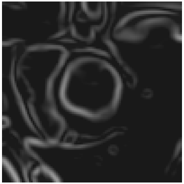

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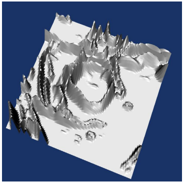

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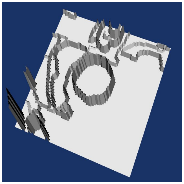

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Hypothesis

- Most existing approaches consider a grayscale image as a vertex-weighted graph

Problem

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- *Watersheds in edge-weighted graphs?*

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- *What mathematical properties?*

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Problem

- *Watersheds in edge-weighted graphs?*
- *What mathematical properties?*
- *How to efficiently compute them?*

1. Watershed cuts: definition and consistency
2. Relative minimum spanning forests: watershed optimality
3. Algorithm

Edge-weighted graph

- Let $G = (V, E)$ be a graph.
- Let F be a map from E to $\mathbb{Z}$.

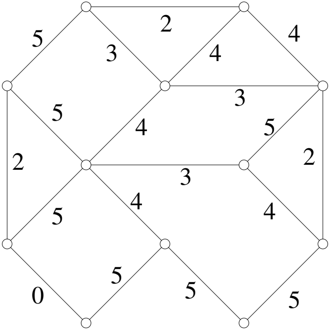
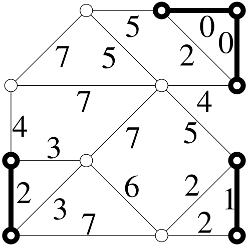


Image and edge-weighted graph

For applications to image analysis

- V is the set of *pixels*
- E corresponds to an *adjacency relation* on V, (*e.g.*, 4- or 8-adjacency in 2D)
- The altitude of u, an edge between two pixels x and y, represents the *dissimilarity between x and y*
    - $F(u) = |I(x) - I(y)|$.

Regional minima

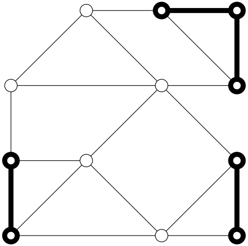


Definition

A subgraph X of G is a minimum of F (at altitude k) *if:*

- *X is connected; and*
- *k is the altitude of any edge of X; and*
- *the altitude of any edge adjacent to X is strictly greater than k*

Extension



a subgraph X

Definition (from Def. 12, (Ber05))

Let X and Y be two non-empty subgraphs of G.

- *We say that Y is an extension of X (in G) if $X \subseteq Y$ and if any component of Y contains exactly one component of X.*

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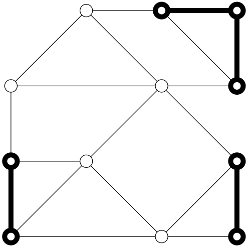

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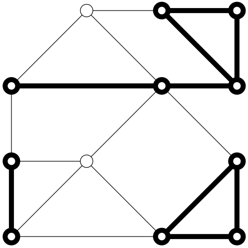

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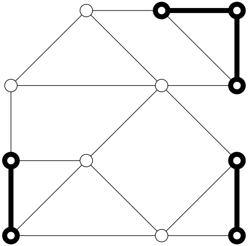

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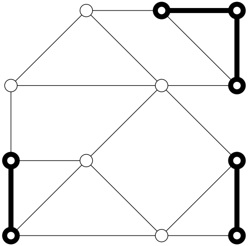

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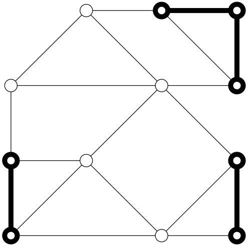

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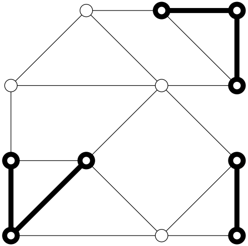

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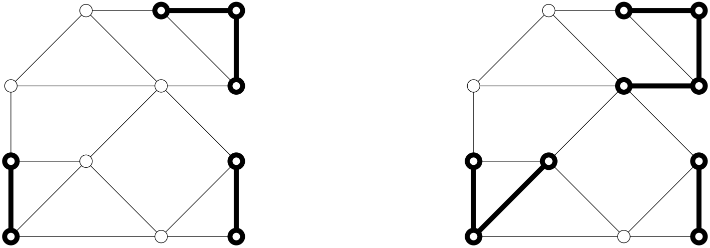

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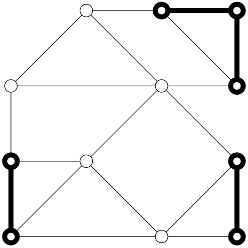

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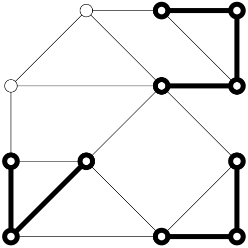

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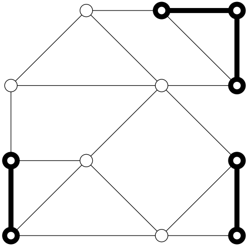

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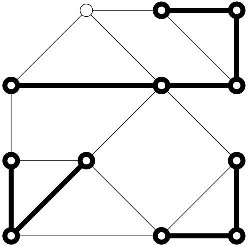

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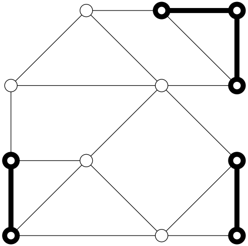

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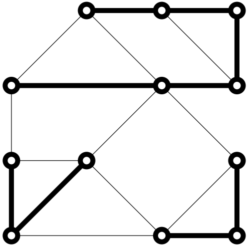

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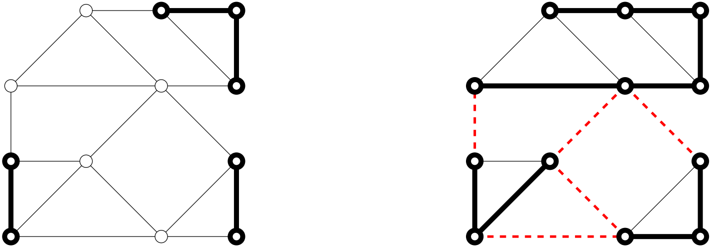
Graph cut



a subgraph X — a *(graph) cut* S for X

Definition (Graph cut)

Let X be a subgraph of G and $S \subseteq E$, a set of edges.

- *We say that S is a (graph) cut for X if $\overline{S}$ is an extension of X and if S is minimal for this property*

Watershed cut

- *The church of Sorbier*

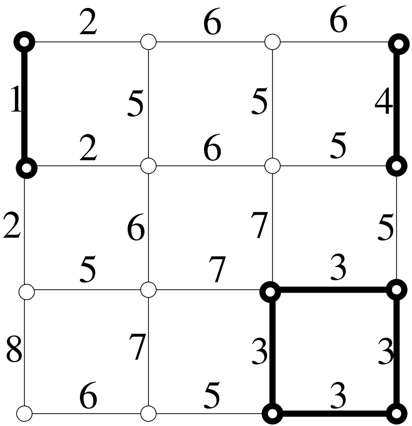
  (a topographic intuition)

Definition (drop of water principle)

The set $S \subseteq E$ is a watershed cut of F if $\overline{S}$ is an extension of $M(F)$ and if for any $u = \{x_0, y_0\} \in S$, there exist $\langle x_0, \ldots, x_n \rangle$ and $\langle y_0, \ldots, y_m \rangle$, two descending paths in $\overline{S}$ such that:

1. *x_n and y_m are vertices of two distinct minima of F; and*
2. *$F(u) \geq F(\{x_0, x_1\})$ if $n > 0$ and $F(u) \geq F(\{y_0, y_1\})$ if $m > 0$*

Watershed cut: example



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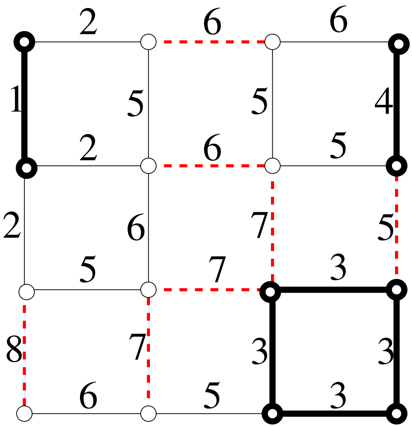

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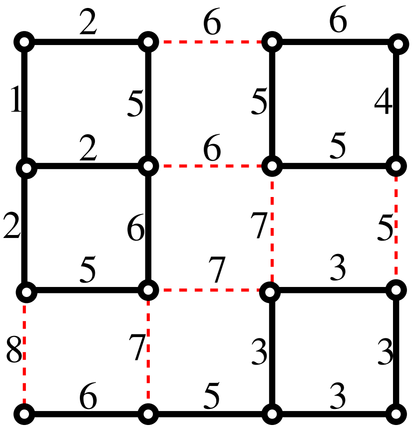

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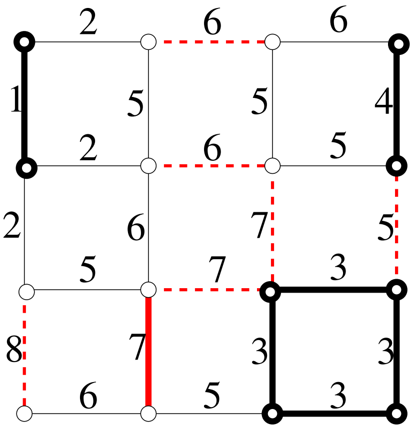

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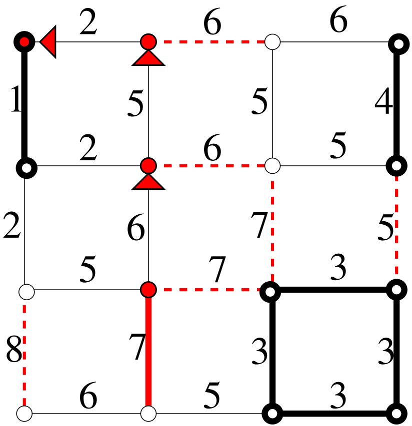

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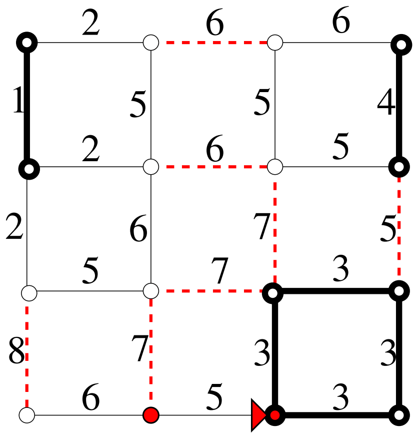
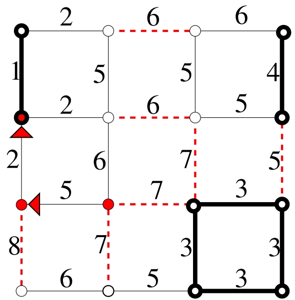


Steepest descent



Definition

Let $\pi = \langle x_0, \ldots, x_l \rangle$ be a path in G.

- *The path π is a* path with steepest descent for *F if:*
  $\forall i \in [1, l]$, $F(\{x_{i-1}, x_i\}) = \min_{\{x_{i-1}, y\} \in E} F(\{x_{i-1}, y\})$

Catchment basins by a steepest descent property

Definition

Let S be a cut for $M(F)$, the minima of F.
We say that S is a basin cut of F *if, from each point of V to $M(F)$, there exists, in the graph induced by $\overline{S}$, a path with steepest descent for F.*

Catchment basins by a steepest descent property

Theorem (consistency)

An edge-set $S \subseteq E$ is a basin cut of F if and only if *S is a watershed cut of F.*

Relative forests

In 1994, F. Meyer shows the links between flooding from markers and minimum spanning forest

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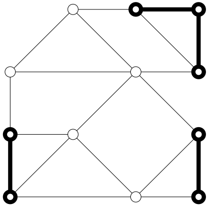
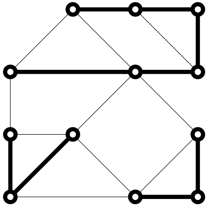
Problem

- *What about watershed cuts ?*

Relative forest



a subgraph X



a *forest Y relative to X*

Definition

Let X and Y be two non-empty subgraphs of G.
We say that Y is a forest relative to X *if:*

1. *Y is an extension of X; and*
2. *any cycle of Y is also a cycle of X*

Minimum spanning forest

- The *weight of a forest* Y is the sum of its edge weights *i.e.*, $\sum_{u \in E(Y)} F(u)$.

Minimum spanning forest

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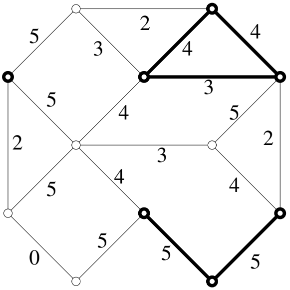
Definition

We say that Y is a minimum spanning forest (MSF) relative to X *if* Y *is a spanning forest relative to* X *and if the weight of* Y *is less than or equal to the weight of any other spanning forest relative to* X.

Minimum spanning forest: example



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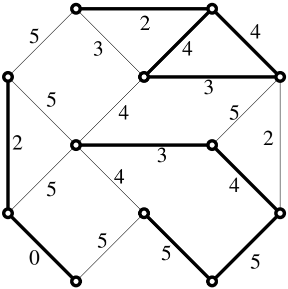

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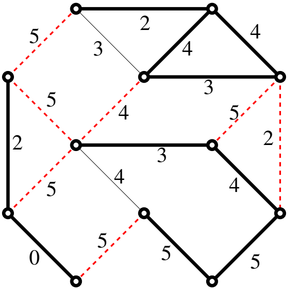

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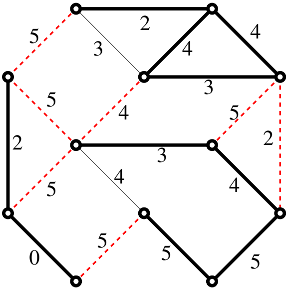
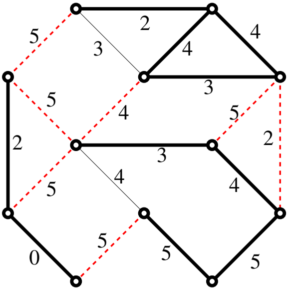


- If Y is a MSF relative to X, there exists a unique cut S for Y and this cut is also a cut for X;

Minimum spanning forest: example



- If Y is a MSF relative to X, there exists a unique cut S for Y and this cut is also a cut for X;
- In this case, we say that S is a *MSF cut for X*.

Watershed optimality

Theorem

An edge-set $S \subseteq E$ is a MSF cut for the minima of F if and only if *S is a watershed cut of F.*

Minimum spanning tree

- Computing a MSF $\Leftrightarrow$ computing a minimum spanning tree

Minimum spanning tree

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- Best algorithm [CHAZEL00]: quasi-linear time

Minimum spanning tree

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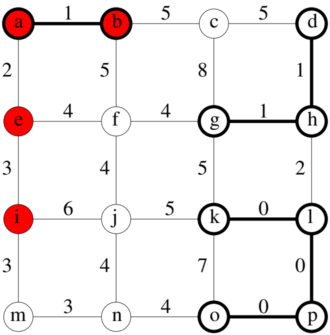
Problem

Can we reach a better complexity for computing watershed cuts?

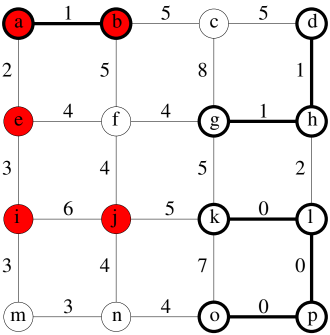
Streams



Definition

We say that a vertex-set $L \subseteq V$ is a stream *if, for any two points x and y of L, there exists, in L, either a path from x to y or from y to x, with steepest descent for F*

Streams



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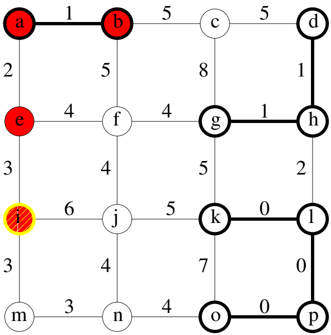
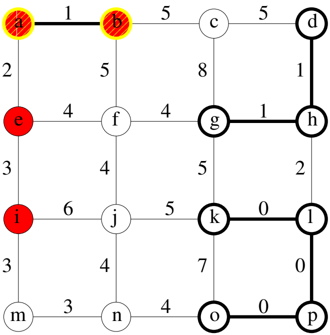


Definition

We say that a point x of a stream L is a top of L if for any point y in L, there exists, from x to y, a path in L, which is a path with steepest descent for F

Streams



Definition

We say that a point x of a stream L is a bottom of L *if for any point y in L, there exists, from y to x, a path in L, which is a path with steepest descent for F*

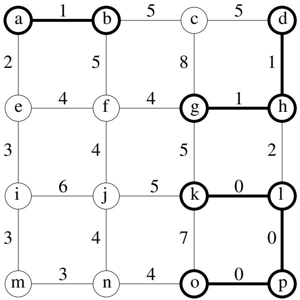
Streams

Problem

How to extract a stream ?

Streams concatenation

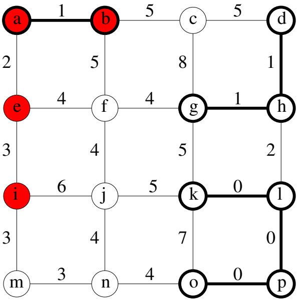
Stream concatenation



Definition

Let L_1 and L_2 be two disjoint streams ($L_1 \cap L_2 = \emptyset$).
We say that L_1 is under L_2 *(written $L_1 \prec L_2$), if there exist a top x of L_1, a bottom y of L_2, such that x and y are adjacent and $\langle x, y \rangle$ is a path with steepest descent for F*

Stream concatenation



A stream L_1

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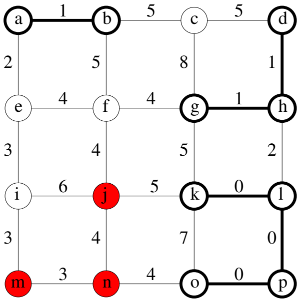
Stream concatenation



A stream L_2

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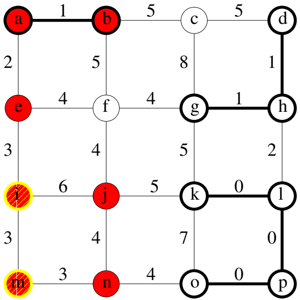

$L_1 \prec L_2$

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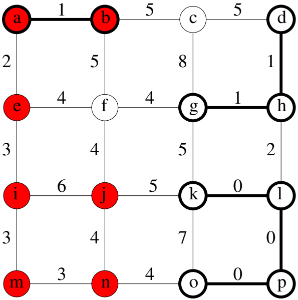
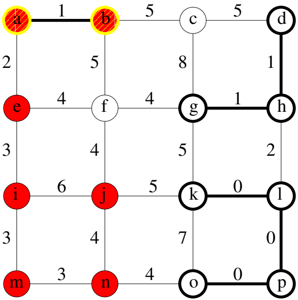


$L = L_1 \cup L_2$ is also a stream

Property

Let L_1 and L_2 be two disjoint streams ($L_1 \cap L_2 = \emptyset$).
If $L_1 \prec L_2$, then $L_1 \cup L_2$ is a stream

Stream concatenation

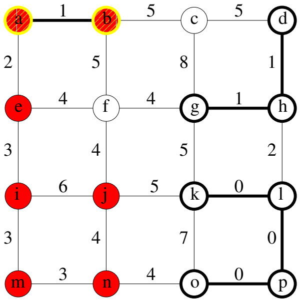


The bottoms of L

Definition

A stream L is an $\prec$-stream if there is no stream under L

Stream concatenation



L is also an $\prec$-stream

Definition

A stream L is an $\prec$-stream *if there is no stream under L*

Streams and minima

- The proposed algorithm is based on $\prec$-streams extraction

Streams and minima

Property

A stream L is an $\prec$-stream if and only if *L contains the vertex set of a minimum of F*

Problem

From the streams of F, how can we partition the vertex set of G?

Flow familly

- Let $\mathcal{L} = \{L_1, \ldots, L_n\}$ be a set of n $\prec$-streams. We say that $\mathcal{L}$ is a *flow family* if:
  - $\cup\{L_i \mid i \in \{1, \ldots, n\}\} = V$; and
  - for any two distinct L_1 and L_2 in $\mathcal{L}$, if $L_1 \cap L_2 \neq \emptyset$, then $L_1 \cap L_2$ is the vertex set of a minimum of F.

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- We remark that a flow family induces a unique graph cut for the minima of F.

Streams and watershed cut

Theorem

An edge set $S \subseteq E$ is a watershed cut of F if and only if *S is a cut induced by a flow family.*

Linear-time algorithm

Algorithm 1: LPE par flux

Data: (V, E, F): an edge-weighted graph;

Result: ψ: a flow mapping of F.

foreach $x \in V$ **do** $\psi(x) := NO\_LABEL$;
$nb\_labs := 0$; /* the number of minima already found */
foreach $x \in V$ *such that* $\psi(x) = NO\_LABEL$ **do**
    $[L, lab] := \texttt{stream}(V, E, F, \psi, x)$;
    if $lab = \emptyset$ **then** /* L is an $\prec$-stream */
        $nb\_labs{+}{+}$;
        foreach $y \in L$ **do** $\psi(y) := nb\_labs$;
    else
        foreach $y \in L$ **do** $\psi(y) := lab$;

Linear-time algorithm

Function `Stream(` V, E, F, ψ, x `)`

Data: (V, E, F): an edge-weighted graph; ψ: a labeling of V; x: a point in V.

Result: $[L, lab]$ where L is a stream such that x is a top of L, and *lab* is either a label of an $\prec$-stream under L or $\emptyset$.

$L := \{x\}$;
$L' := \{x\}$; /* the set of non-explored bottoms of L */
while *there exists* $y \in L'$ **do**
    $L' := L' \setminus \{y\}$;
    $breadth\_first := TRUE$;
    while *(breadth_first) and (there exists* $\{y, z\} \in E$ *such that* $z \notin L$ *and* $F(\{y, z\}) = F(y)$*)* **do**
        if $\psi(z) \neq NO\_LABEL$ **then**
            /* there is an $\prec$-stream under L already labelled */
            return $[L, \psi(z)]$;
        else if $F(z) < F(y)$ **then**
            $L := L \cup \{z\}$; /* z is now the only bottom of L */
            $L' := \{z\}$; /* hence, switch to depth-first exploration */
            $breadth\_first := FALSE$;
        else
            $L := L \cup \{z\}$; /* $F(z) = F(y)$, thus z is also a bottom of L */
            $L' := L' \cup \{z\}$; /* continue breadth-first exploration */
return $[L, \emptyset]$;

Algorithm

Result

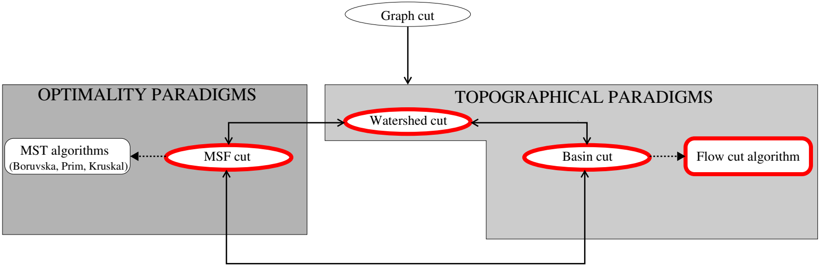
- *Stream Algorithm runs in linear time whatever the range of the input map*
  - *No need to sort*
  - *No need to use a hierarchical queue*

Algorithm

Result

- *Stream Algorithm runs in linear time whatever the range of the input map*
  - *No need to sort*
  - *No need to use a hierarchical queue*
- *Furthermore, Stream Algorithm does not need to compute the minima as a pre-processing step.*

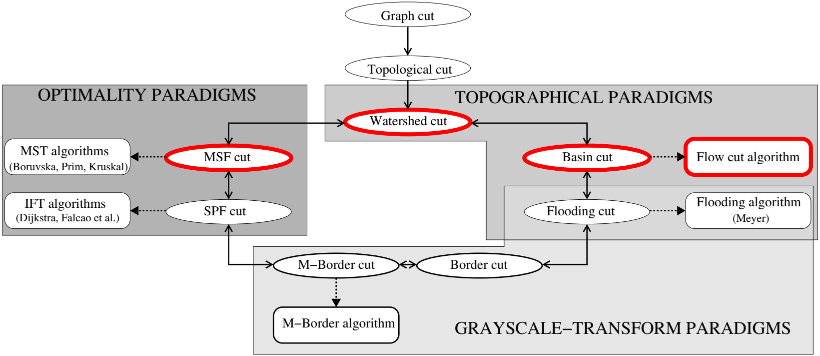
Conclusion


Graph cut
OPTIMALITY PARADIGMS
TOPOGRAPHICAL PARADIGMS
Watershed cut
MST algorithms
(Boruvska, Prim, Kruskal)
MSF cut
Basin cut
Flow cut algorithm

Conclusion

- In fact, there is more to say on watershed cuts . . .

Conclusion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(Dijkstra, Falcao et al.)
SPF cut
Flooding cut
Flooding algorithm
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M−Border cut
Border cut
M−Border algorithm
GRAYSCALE−TRANSFORM PARADIGMS

Perspectives

- Hierarchical segmentations
  - Saliency of watershed contours
  - Incremental MSF

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  - Saliency of watershed contours
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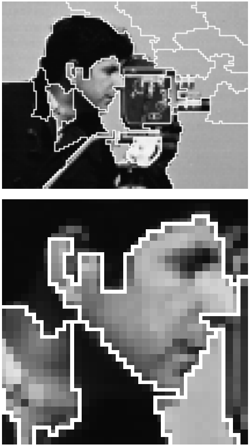
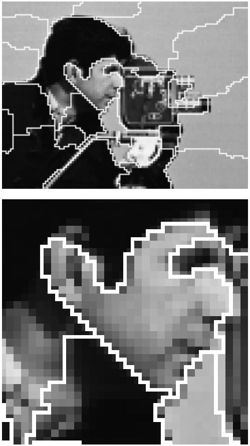
- Topological properties of watershed cuts (simplicial, cubical complexes)
- Minimum spanning tree by watersheds

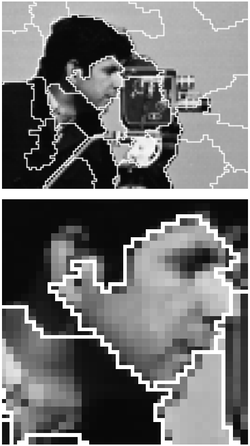
To finish : illustration



Edges

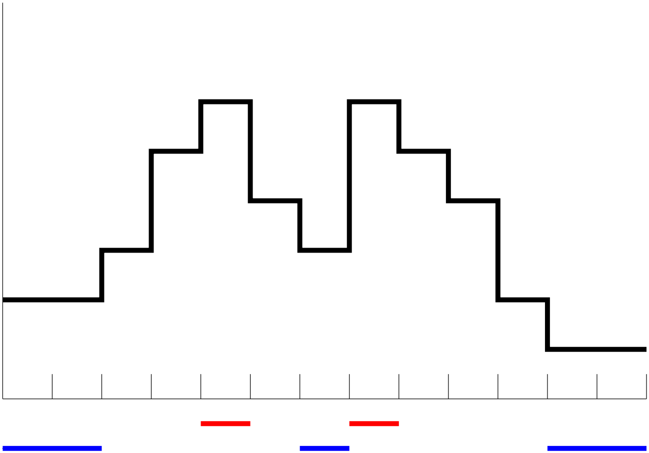


Vertices+Deriche



Vertices+morpho. grad.

The plateau problem



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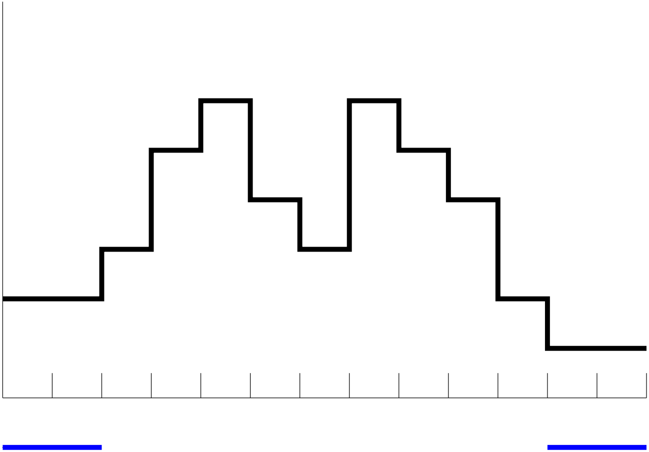

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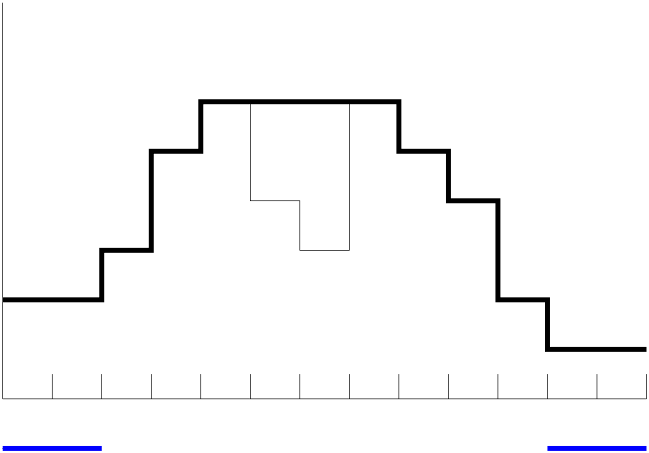

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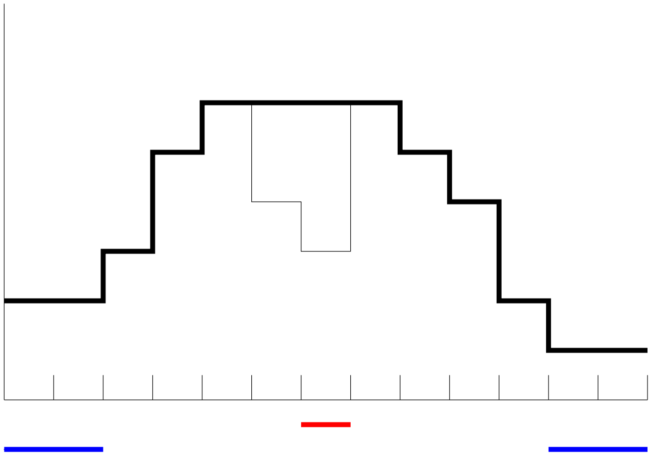

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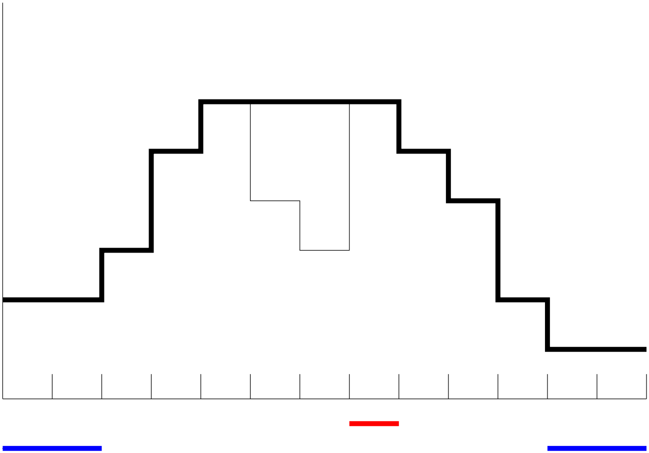

The plateau problem