Adjacency Stable Connected Operators and Set Levelings

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Outline

Part I

- Background I

  Adjacency stable connected operators

  Levelings and set levelings

- Relationships between adjacency stable connected operators and set levelings
- Some useful properties

Part II

- Background II: strong property
- Some objections to statements about the strong property of levelings in the literature
- A commutative property for alternated attribute filters

Conclusion

PART I

BACKGROUND

- Mathematical Morphology
- We will focus on the set or binary framework
- Basic morphological filters

  Openings

  Closings

- Connectivity

  Connected class

  A connected class $\mathcal{C}$ in $\mathcal{P}(E)$ is a subset of $\mathcal{P}(E)$ such that

  (a) $\emptyset \in \mathcal{C}$ and for all $x \in E$, $\{x\} \in \mathcal{C}$; and

  (b) for each family C_i in $\mathcal{C}$, $\bigwedge_i C_i \neq \emptyset$ implies $\bigvee_i C_i \in \mathcal{C}$.

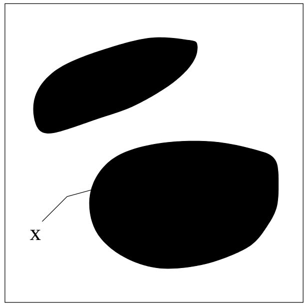
BACKGROUND (CONT.)

- Opening γ_x

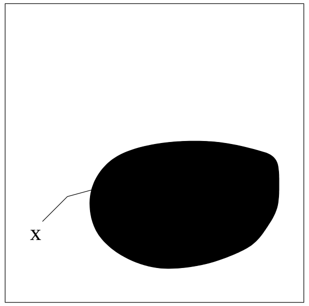
  The subclass $\mathcal{C}_x$ that has all members of $\mathcal{C}$ that contain x (i.e., $\mathcal{C}_x = \{C \in \mathcal{C} : x \in C\}$) leads to the definition of an opening γ_x called *point opening*. For all $x \in E$, $A \in P(E)$,

$$\gamma_x(A) = \bigvee \{C : C \in \mathcal{C}_x, C \leq A\}. \tag{1}$$

  Connected component extraction operation



(a) Input set A



(b) $\gamma_x(A)$

BACKGROUND (CONT.)

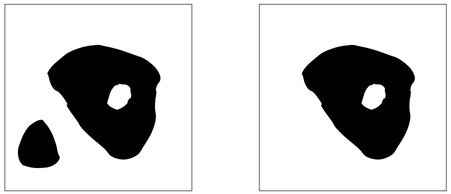
- Connected operator

  Relationships between the flat zones of the input and the output

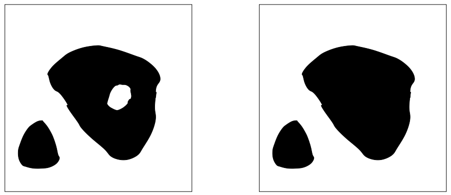
    Inclusion relationship

- Connected openings and connected closings

  Connected opening:



  Connected closing:

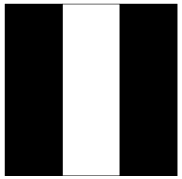


- Connectivity requirement: strong connectivity

Adjacency stable connected operators

- Origin of the concept: consideration of the following question:

  Given an input set, there are several connected outputs that satisfy the inclusion relationship between the input and the output.

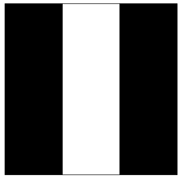


Input set

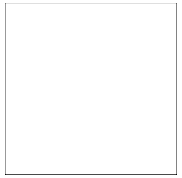
8 possible connected outputs

However, some possible outputs **cannot** be computed using morphological connected openings and closings.

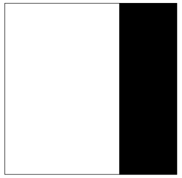
ADJACENCY STABLE CONNECTED OPERATORS (CONT.)



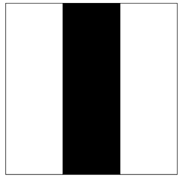
(a) Input set



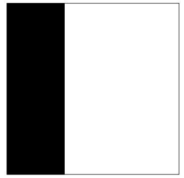
(b) √



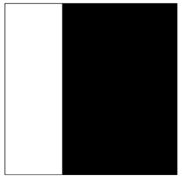
(c) √



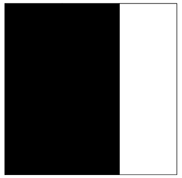
(d) **Not possible**



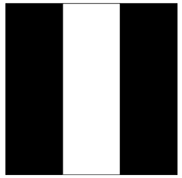
(e) √



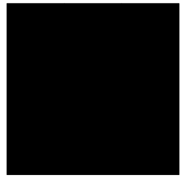
(f) **Not possible**



(g) **Not possible**



(h) √



(i) √

- Normal morphological connected operators impose certain adjacency restrictions.

ADJACENCY STABLE CONNECTED OPERATORS (CONT.)

Definition 1 *Let E be a space endowed with γ_x, $x \in E$. An operator $\psi : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ is* **adjacency stable** *if, for all $x \in E$:*

$$\gamma_x(\mathrm{id} \bigvee \psi) = \gamma_x \bigvee \gamma_x \psi. \tag{2}$$

[Crespo-Serra-Schafer, 1993] [Crespo, 1993] [Crespo and Schafer, 1997].

Note that γ_x commutes under the inf ($\gamma_x(\bigwedge_i \psi_i) = \bigwedge_i \gamma_x \psi_i$) but not in general under the sup.

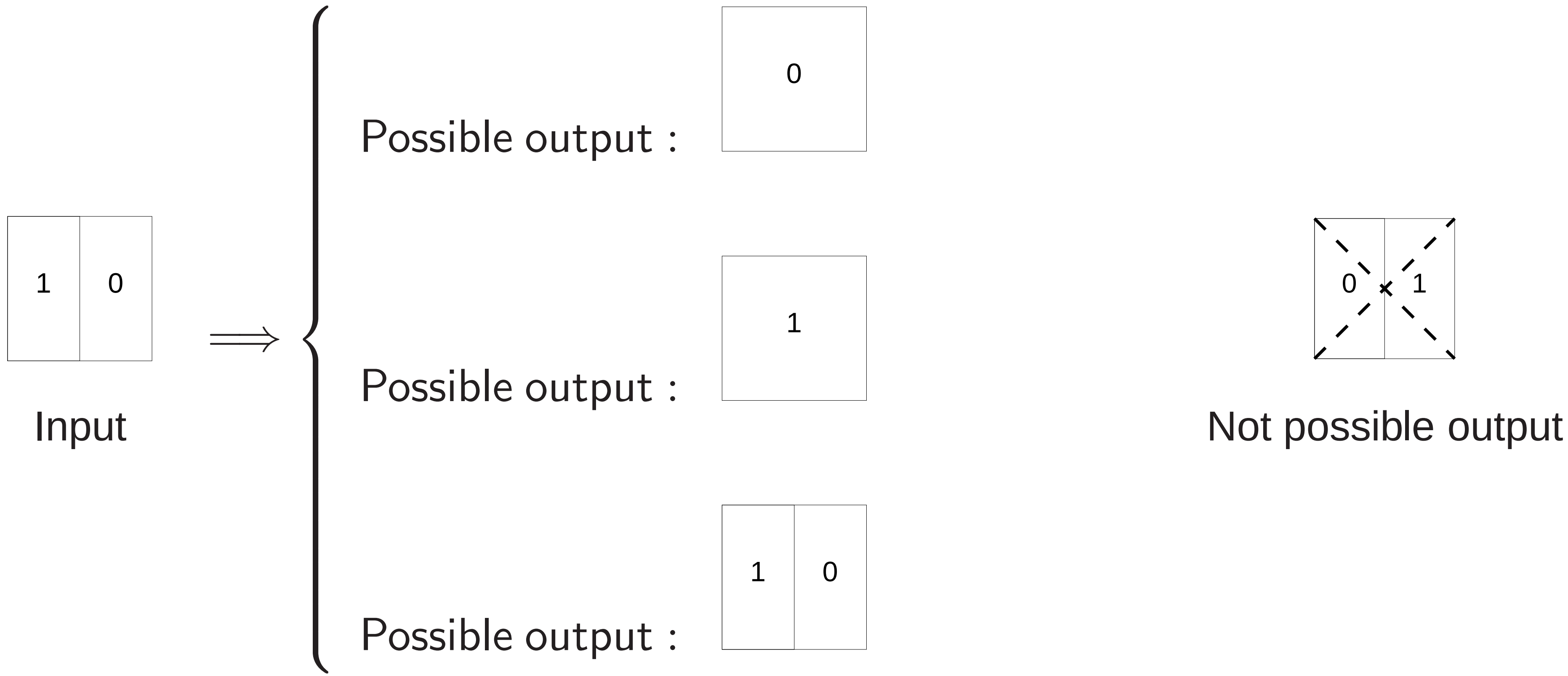
Both grains and pores are considered.

What matters is the switch from grain to pore and vice-versa.

The dual of an adjacency stable operator is adjacency stable.

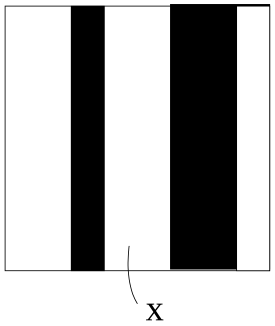
Adjacency stable connected operators (cont.)

For an adjacency stable connected operator:

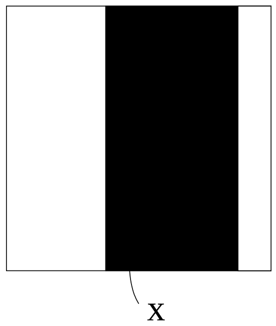


ADJACENCY STABLE CONNECTED OPERATORS (CONT.)

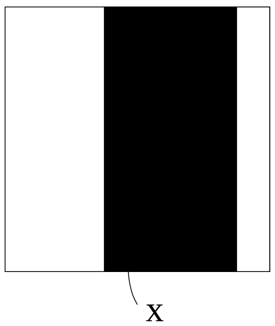
Example in which the adjacency stability equation is not satisfied



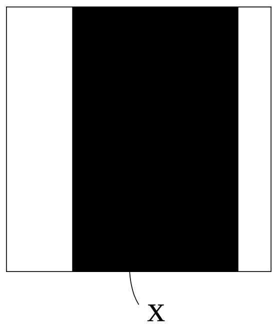
(a) Input set A (in dark)



(b) $\psi(A)$ (ψ is adjacency unstable)

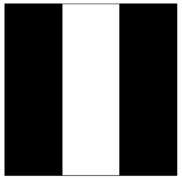


(c) $\gamma_x(A) \bigvee \gamma_x\psi(A)$

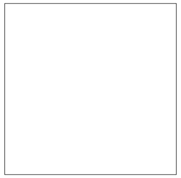


(d) $\gamma_x(\mathrm{id} \bigvee \gamma_x\psi)(A)$

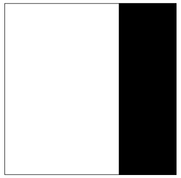
ADJACENCY STABLE CONNECTED OPERATORS (CONT.)



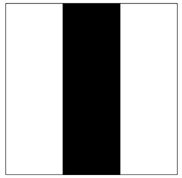
(a) Input set (in dark)



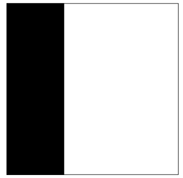
(b)



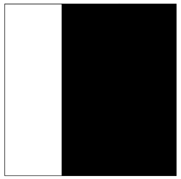
(c)



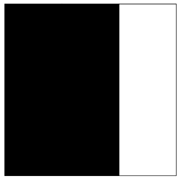
(d) **No AS**



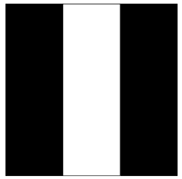
(e)



(f) **No AS**



(g) **No AS**



(h)

(i)

Levelings

Definition 2 *An image g is a leveling of an input image f if and only if:*

$$\forall\ (p, q) \text{ neighboring pixels} : g_p > g_q \ \Rightarrow\ f_p \geq g_p \text{ and } g_q \geq f_q \tag{3}$$

Notes:

The previous definition of leveling is that in [Meyer, 1998a, Definition 4 (p. 193)] [Meyer and Maragos, 2000, Definition 2.2 (p. 4)].

We focus on set operators.

- Let us write expression (3) as:

$$\forall\ (p, q) \text{ neighboring pixels} : I'_p > I'_q \ \Rightarrow\ I_p \geq I'_p \text{ and } I'_q \geq I_q$$

where I and I' denote, respectively, the input and output images.

Set levelings

- Set levelings are levelings in the set or binary framework.

  Binary function expressions will be used in the following.

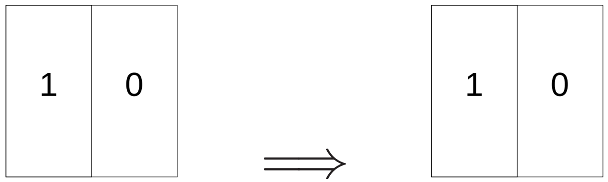
- An inequality such as $I'_p > I'_q$ can only occur when there is a discontinuity where I'_p and I'_q are 1 and 0, respectively. Then, the leveling expression

$$I'_p > I'_q \ \Rightarrow \ I_p \geq I'_p \text{ and } I'_q \geq I_q$$

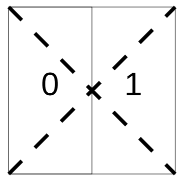
reduces to

$$1 > 0 \ \Rightarrow \ 1 \geq 1 \text{ and } 0 \geq 0 \tag{4}$$

I.e., I_p has to be 1, and I_q must be 0.



Output Input



- If $I'_p = 1$ and $I'_q = 0$, the case where $I_p = 0$ and I_q = 1 is excluded:

Relationships between adjacency stable connected operators and set levelings

- Relationships:
  (a) Both adjacent stable connected operators and set levelings impose restrictions on the input/output variations.
  (b) The imposed restrictions are equivalent

  If $I'_p = 1$ and $I'_q = 0$, then I_p and I_q must be 1 and 0, respectively.

- Thus: **the set leveling concept and the adjacency stable connected operator concept are equivalent**.

- Chronology of the introduction of concepts

  [Crespo-Serra-Schafer, 1993] [Crespo, 1993] [Crespo and Schafer, 1997] are prior in time to [Meyer, 1998a] [Meyer, 1998b] [Meyer and Maragos, 2000] [Meyer, 2004].

Some comments about the gray-level case

- We can directly extend those adjacency restrictions to flat gray-level connected operators that commute with thresholding.

  The restrictions must hold for all sections of the input and output gray-level functions.

  If p and q are neighbors to each other (or if they belong to adjacent flat zones), then:

$$
\begin{array}{lccc}
\text{(a)} & I_p = I_q & \Longrightarrow & I'_p = I'_q . \\
\text{(b)} & I_p > I_q & \Longrightarrow & \left\{ \begin{array}{c} I'_p > I'_q \\ \text{or} \\ I'_p = I'_q \end{array} \right.
\end{array}
\tag{5}
$$

  Note: the case symmetric to (b) is not shown.
  The case ruled out is: $I_p < I_q$, and $I'_p > I'_q$ (as well as the symmetric one: $I_p > I_q$, and $I'_p < I'_q$).

  This case is also excluded by the expression of levelings.

- Increasingness requirement

Some useful properties

Some results about adjacency stable connected operators and set levelings:

Property 1 *Extensive and anti-extensive mappings are adjacency stable.*

Property 2 *The class of adjacency stable connected operators is closed under the sup, the inf and the sequential composition operations.*

Lemma 1 *Let E be a space endowed with γ_x, $x \in E$. A connected operator $\psi : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ is adjacency stable if and only if, for all $A \in \mathcal{P}(E)$, $\psi(A)$ and $A \setminus \psi(A)$ are not connected to each other (i.e., are not adjacent).*

Lemma 1 is useful to relate the input and the output.

Note: see also [Crespo-Serra-Schafer, 1993] [Crespo, 1993] [Crespo and Schafer, 1997].

Adjacency stability and connected-component locality

Part II

BACKGROUND II: STRONG PROPERTY

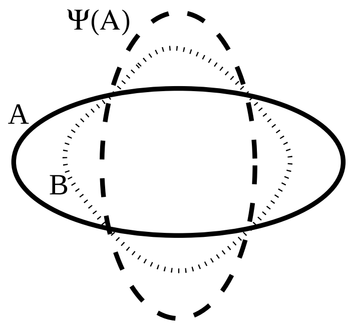
- A morphological filter Ψ is strong if and only if

$$\Psi \;=\; \Psi(\mathrm{id} \bigwedge \Psi) \;=\; \Psi(\mathrm{id} \bigvee \Psi). \tag{6}$$

We also have that if Ψ is a strong filter:

$$A \bigwedge \Psi(A) \leq B \leq A \bigvee \Psi(A) \Longrightarrow \Psi(A) = \Psi(B) \tag{7}$$

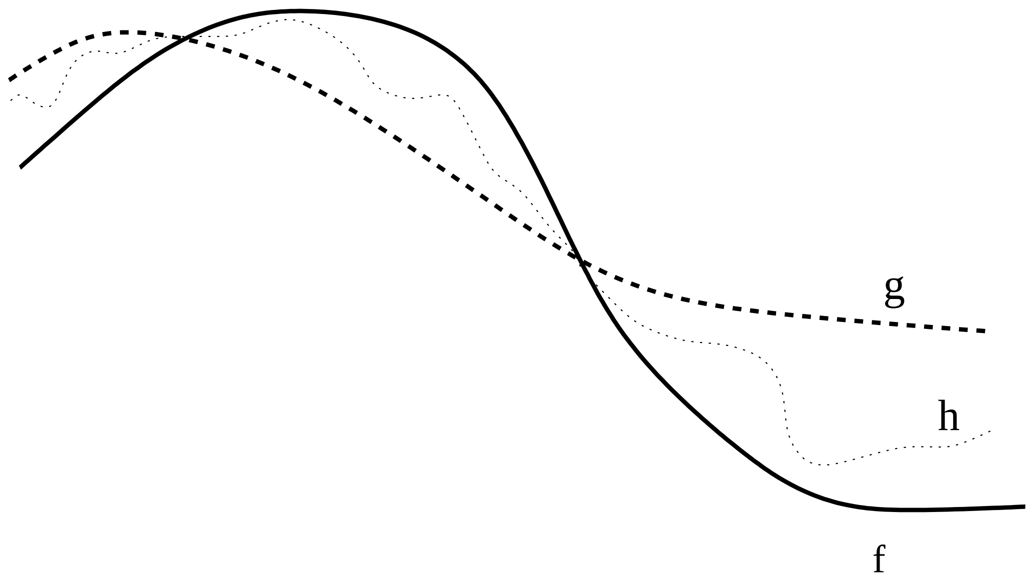
- Binary example:



Sets A, B y $\Psi(A)$: If Ψ is a strong filter, then $\Psi(B) = \Psi(A)$.

BACKGROUND II: STRONG PROPERTY (CONT.)

- 1-D non-binary example:



If f is an input function and $g = \Psi(f)$, where Ψ is a strong filter, it is true that $\Psi(h) = \Psi(f) = g$ for all function h between f and g.

Background II: strong property (cont.)

- If an operator can be expressed as a sequential composition of an opening and a closing, and vice-versa, then it is a strong filter.

Some marker-based operators and the strong property

- Some objections to statements regarding levelings and the strong property in some previous research works can be made.

- Let us define two operators, $\overline{\gamma}$ and $\underline{\varphi}$, based on makers that are presented in [Serra, 2000, p. 176].

  Besides one normal input set, those operators use a second one, which is a marker set.

Definition 3 *Let A and M be two sets. The connected operator $\overline{\gamma}$ of a set A based on marker M, symbolized by $\overline{\gamma}(A, M)$, is defined as:*

$$\overline{\gamma}(A, M) = \bigcup \{\gamma_x(A) : \gamma_x(A) \parallel M\} \tag{8}$$

Definition 4 *Let A and B be, respectively, a grain (a connected set) and a set. $A \parallel B$ if A and B have a non-empty intersection or are adjacent. [Serra, 2000, Definition 7.3]*

SOME MARKER-BASED OPERATORS AND THE STRONG PROPERTY (CONT.)

Definition 5 *Let A and M be two sets. The connected operator $\underline{\varphi}$ of a set A based on marker N, symbolized by $\underline{\varphi}(A, N)$, is defined as:*

$$\complement[\underline{\varphi}(A, \complement N)] = \bigcup \{\gamma_x(\complement A), x \in E : \gamma_x(\complement A) \parallel N\} \tag{9}$$

- In [Serra, 2000], it is established that there exists a commutative property for $\overline{\gamma}$ and $\underline{\varphi}$ ([Serra, 2000, Theorem 7.3]):

$$\overline{\gamma}(\underline{\varphi}(A, \complement N), M) = \underline{\varphi}(\overline{\gamma}(A, M), \complement N) \tag{10}$$

- It is indicated in [Serra, 2000] that $\overline{\gamma}(\underline{\varphi}(A, \complement N), M)$ (or $\underline{\varphi}(\overline{\gamma}(A, M), \complement N)$) is a leveling and that is a strong filter.

  Expression (10) is considered in [Serra, 2000] as a sequential composition of an opening and a closing, and of a closing and an opening.

Some marker-based operators and the strong property (cont.)

- Moreover, in [Meyer and Maragos, 2000] [Meyer, 2004], it is indicated that levelings are strong filters.

  The discussion refers to a commutative expression analogous to the aforementioned one.

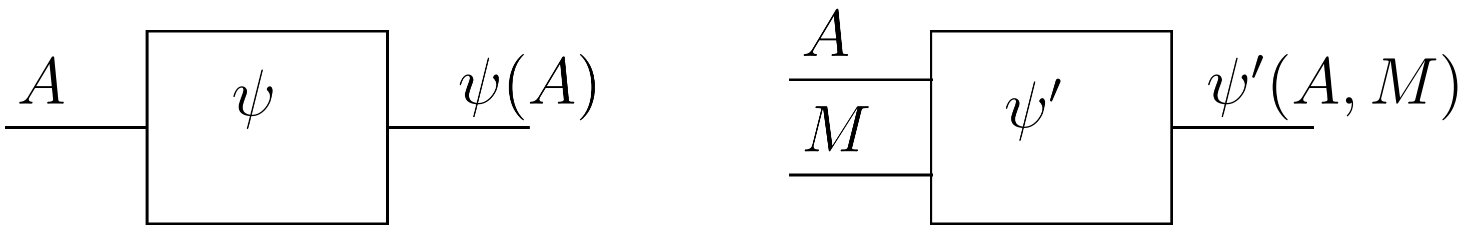
- This should be clarified:

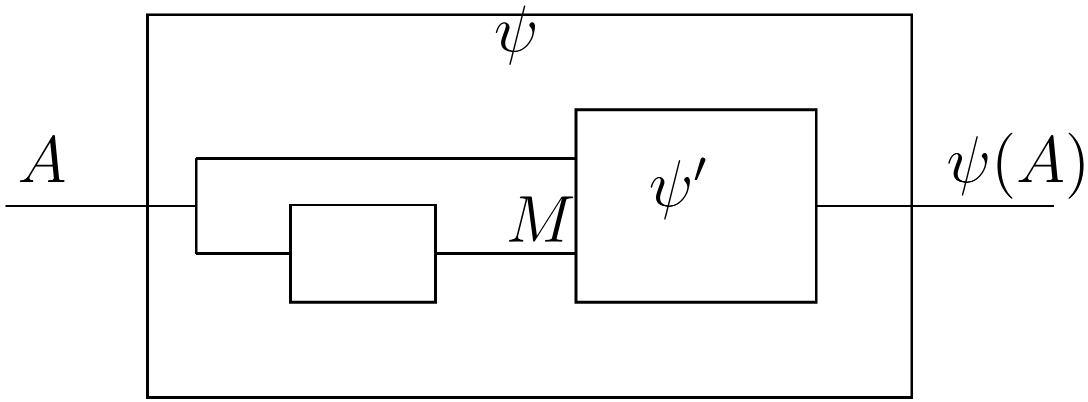
  <u>Not all levelings are strong filters.</u>

- It seems there could be some confusion about whether all levelings can be formulated as sequential compositions of an opening and a closing, and vice-versa.

SOME MARKER-BASED OPERATORS AND THE STRONG PROPERTY (CONT.)

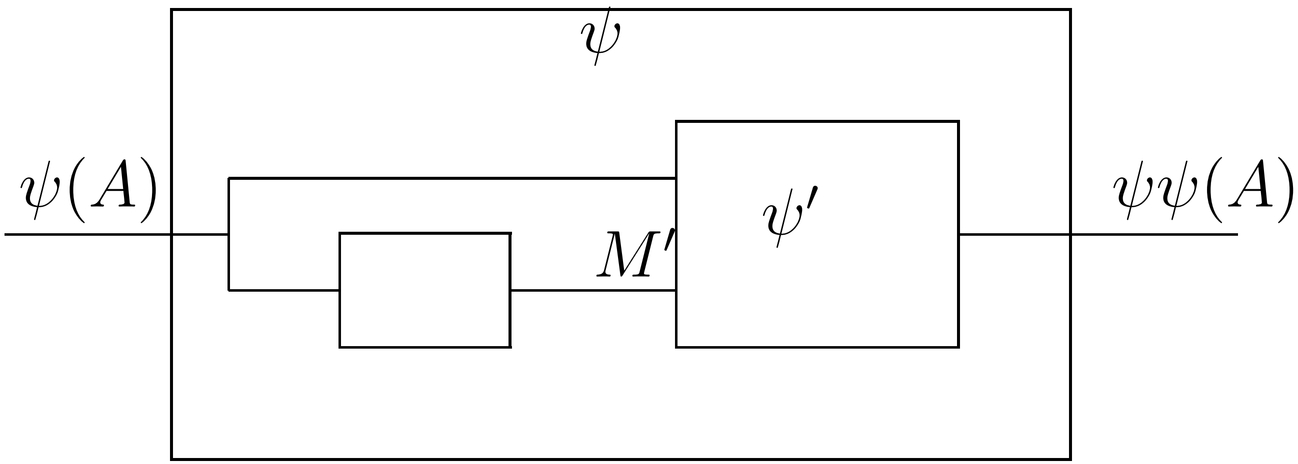
- Operators with markers





Some marker-based operators and the strong property (cont.)

- Operators with markers (cont.)



- See [Crespo et al., 2002] [Crespo and Maojo, 2007] for further discussion about the strong property of connected alternated filters.

A COMMUTATIVE PROPERTY FOR ALTERNATED ATTRIBUTE FILTERS

- Let $\tilde{\gamma}$ and $\tilde{\varphi}$ denote, respectively, an *attribute opening* [Serra, 1988] and an *attribute closing*.

  Area openings and closings are examples of attribute openings and closings, respectively.

- Alternated attribute filters $\tilde{\varphi}\tilde{\gamma}$ are strong filters.

  Note: in the following, when we write $\tilde{\varphi}\tilde{\gamma}$ it is clear that the criterion (and associated marker) of $\tilde{\varphi}$ is applied to the output computed by the previous $\tilde{\gamma}$.

A COMMUTATIVE PROPERTY FOR ALTERNATED ATTRIBUTE FILTERS (CONT.)

Property 3 *An attribute alternated filter $\tilde{\varphi}\tilde{\gamma}$ can be expressed as a commutative sequential composition of an opening and a closing as follows:*

$$\tilde{\varphi}\tilde{\gamma} = \tilde{\gamma}\,(\mathrm{id} \bigvee \tilde{\varphi}\tilde{\gamma}) = (\mathrm{id} \bigvee \tilde{\varphi}\tilde{\gamma})\,\tilde{\gamma} \tag{11}$$

Proof of Property 3: There are two equalities to consider.

(a) Let A be a set. Since $\tilde{\gamma}$ is connected-component local we have $\tilde{\gamma} = \bigvee_x \gamma_x \tilde{\gamma} = \bigvee_x \tilde{\gamma}\gamma_x$. Thus, $\tilde{\gamma}(\mathrm{id} \bigvee \tilde{\varphi}\tilde{\gamma})(A) = \bigvee_x \gamma_x \tilde{\gamma}(\mathrm{id} \bigvee \tilde{\varphi}\tilde{\gamma})(A) = \bigvee_x \tilde{\gamma}\gamma_x(\mathrm{id} \bigvee \tilde{\varphi}\tilde{\gamma})(A)$. From Property 1 and Property 2, $\tilde{\varphi}\tilde{\gamma}$ is an adjacency stable connected operator, and, from Lemma 1, $\tilde{\varphi}\tilde{\gamma}(A)$ and $A \setminus \tilde{\varphi}\tilde{\gamma}(A)$ are not adjacent [Crespo, 1993] [Crespo and Schafer, 1997]. Then,

$$\tilde{\gamma}\gamma_x(\mathrm{id} \bigvee \tilde{\varphi}\tilde{\gamma})(A) = \begin{cases} \tilde{\gamma}\gamma_x(A) = \emptyset, & x \in A \setminus \tilde{\varphi}\tilde{\gamma}(A). \\ \tilde{\gamma}\gamma_x\tilde{\varphi}\tilde{\gamma}(A), & x \in \tilde{\varphi}\tilde{\gamma}(A). \end{cases} \tag{12}$$

Thus, $\bigvee_x \tilde{\gamma}\gamma_x\tilde{\varphi}\tilde{\gamma} = \bigvee_x \gamma_x\tilde{\gamma}\tilde{\varphi}\tilde{\gamma} = \tilde{\gamma}\tilde{\varphi}\tilde{\gamma}$. Finally, $\tilde{\gamma}\tilde{\varphi}\tilde{\gamma} = \tilde{\varphi}\tilde{\gamma}$ (since $\tilde{\varphi}\tilde{\gamma} \leq \tilde{\gamma}\tilde{\varphi}$ and $\tilde{\gamma}\tilde{\varphi}\tilde{\gamma} = \tilde{\varphi}\tilde{\gamma}$ [Serra and Salembier, 1993] [Salembier and Serra, 1995]).

A COMMUTATIVE PROPERTY FOR ALTERNATED ATTRIBUTE FILTERS (CONT.)

Proof of Property 3 (cont.):

(b) $(\mathrm{id} \bigvee \tilde{\varphi}\tilde{\gamma})\,\tilde{\gamma} = \tilde{\gamma} \bigvee \tilde{\varphi}\tilde{\gamma}\tilde{\gamma} = \tilde{\gamma} \bigvee \tilde{\varphi}\tilde{\gamma} = \tilde{\varphi}\tilde{\gamma}.$

- Notes:

  (a) $(\mathrm{id} \bigvee \tilde{\varphi}\tilde{\gamma})$ is a closing (and different from $\tilde{\varphi}$).

  (b) Property 3 is different from [Heijmans, 1999, Proposition 10.2].

  (c) This proof also provides an example of using adjacent stable connected operators properties to manipulate expressions.

  (d) Concerning filter expressions and decompositions, see also [Crespo and Maojo, 1998].

CONCLUSION

- This work has focused on adjacency stable connected operators and set levelings.
- A close relationship has been identified.

  The leveling notion in the set or binary framework can be traced back to [Crespo, Serra and Schafer, 1993] [Crespo, 1993] [Crespo and Schafer, 1997].

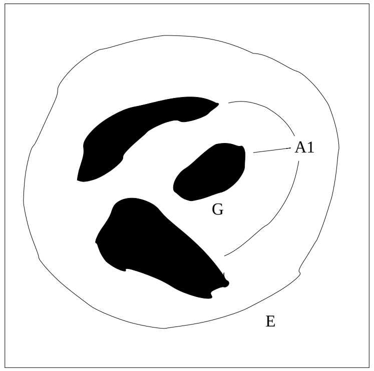
  Some useful properties for manipulating expressions with levelings have been commented.
- Some objections to statements about the strong property for levelings have been raised.
- A commutativity property for alternated attribute filters has been presented.

QUESTIONS ?

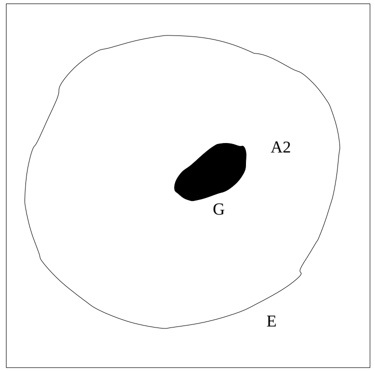
THANKS !

Appendix

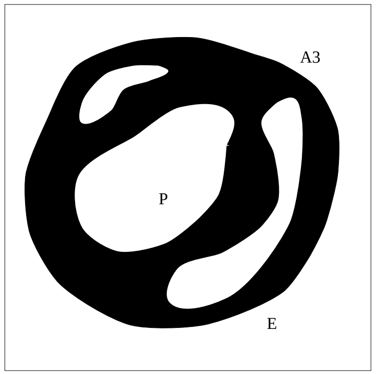
Connected-Component Locality



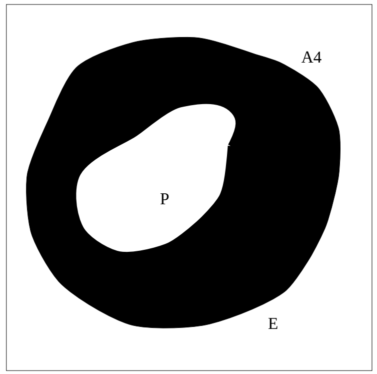
(a) Grain G in set A_1



(b) Grain G in set A_2 ($A_2 = G$)



(c) Pore P in set A_3



(d) Pore P in set A_4 ($A_4 = E \setminus P$)

ATTRIBUTE OPENINGS AND CLOSINGS

- "Basic" connected filters
- Attribute openings and closings can be defined based on the so-called trivial openings and closings. Let T be an increasing criterion.
  - Trivial opening

$$\gamma_{\mathrm{T}}(A) = \begin{cases} A & \text{if } \mathrm{T}(A) = \text{ True} \\ \emptyset & \text{otherwise} \end{cases} \tag{13}$$

  - Trivial closing

$$\varphi_{\mathrm{T}}(A) = \begin{cases} E & \text{if } \mathrm{T}(A) = \text{ True} \\ A & \text{otherwise} \end{cases} \tag{14}$$

- Attribute opening

$$\tilde{\gamma} = \bigvee_{x \in E} \gamma_{\mathrm{T}} \gamma_x, \tag{15}$$

- Attribute closing

$$\tilde{\varphi} = \bigwedge_{x \in E} \varphi_{\mathrm{T}} \varphi_x, \tag{16}$$

Appendix B

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