Helicity modulus behavior of a frustrated XY model

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A frustrated XY model on a square lattice with unequal ferromagnetic and antiferromagnetic bonds is considered. This model can be related, via a Landau-Ginzburg expansion and renormalization-group arguments, to a coupled XY-Ising model. We use this relation to show that, at the Ising transition, the helicity modulus has a logarithmic singularity and its temperature derivative diverge logarithmically. Preliminary results of Monte Carlo simulations in the original model are consistent with this.

Two-dimensional frustrated XY models have been the object of much attention recently. They provide an interesting model with simultaneous $U(1)$ and Z_2 symmetry. The double degeneracy of the ground state, which arises from the competition between ferromagnetic and antiferromagnetic bonds, leads to the possibility of an Ising-like transition in addition to the standard transition of the unfrustrated XY model, due to the continuous symmetry. Some experimental systems, such as a two-dimensional array of Josephson junctions in a perpendicular magnetic field,¹ can actually be modeled by frustrated XY models, and this has motivated much of the recent work in this subject.¹⁻¹²

The possibility of distinct phase transition scenarios has drawn attention to generalized versions of the original fully frustrated XY model.⁸⁻¹¹ One of these models, and the one we will consider here, has been introduced by Berge *et al.*⁸ and consists of XY spins $S_r = (\cos \theta_r, \sin \theta_r)$ interacting by the following Hamiltonian:

$$H = - \sum_{\langle rr' \rangle} J_{rr'} \cos(\theta_r - \theta_{r'}), \quad (1)$$

where the sum is restricted to nearest-neighbor sites. Here $J_{rr'} = J (J > 0)$ on horizontal bonds and $J_{rr'} = J$ and $J_{rr'} = -J\eta$ on alternating columns of the square lattice, as shown in Fig. 1. The fully frustrated XY model is obtained when $\eta = 1$. When $\eta \neq 1$, but greater than $1/3$, Monte Carlo simulations⁸ indicate a double transition with an Ising-like followed by an XY-like transition as temperature increases. The transition seems to coincide when $\eta = 1$. For $\eta < 1/3$, the ground state is not doubly degenerate, and only a XY-like transition takes place. This is in agreement with an analysis based on a Landau-Ginzburg expansion of Eq. (1).⁷ However, other generalized versions,⁹⁻¹¹ which preserve the symmetry of the original model, are consistent with either a single transition or a double transition, but in the reverse sequence to the one mentioned above. This possibility has been suggested before, on the basis of an analysis of the relevant elementary excitations of the system, which consists of fractional vortices in addition to the expected domain wall and integer vortex excitations.^{2,6} Although it has been already recognized that the introduction of unequal ferromagnetic and antiferromagnetic strengths results in a different

symmetry,^{4,7,10,11} the mechanism which leads to different phase transition scenarios is still not completely understood.

The numerical evidence of the double transition for the frustrated XY model described in Eq. (1) appears more clearly in the behavior of the specific heat.⁸ However, in the context of Josephson junction arrays, the measurable quantity is the superfluid density which, in the language of XY spins, corresponds to the helicity modulus.¹³ In this quantity, the XY-like transition appears as a universal jump of $2/\pi$.¹⁴ The Ising transition, however, is associated with a minor effect as indicated in the numerical simulations to be discussed later.¹² It would be of some interest to determine if this effect corresponds to any kind of singularity which could be taken as a signal of the Ising transition.

The purpose of this work is to point out that at the Ising transition the helicity modulus has a logarithmic singularity of the form $(T - T_c) \log |T - T_c|$ and so the Ising transition will appear as a logarithmic divergence of the temperature derivative of the helicity modulus.

To show this, we make use of the result of a previous work,⁷ where the model defined in Eq. (1) is related, via a Landau-Ginzburg expansion, to an effective Hamiltonian in the form of two coupled XY models. By the usual renormalization-group arguments, one expects these models to be in the same universality class. As discussed in Ref. 7, the latter renormalizes to the strong coupling limit. In this limit, it reduces the following coupled XY-Ising model:

$$\frac{H}{kT} = - \sum_{r,\mu} \cos(\theta_r - \theta_{r+\mu}) (A_\mu + B_\mu S_r S_{r+\mu}), \quad (2)$$

FIG. 1. Generalized version of the fully frustrated XY model. Double lines are antiferromagnetic bonds of strength $-J\eta$ and single lines are ferromagnetic bonds.

where $\mu = \hat{x}, \hat{y}$. The couplings A_μ, B_μ are in general distinct and anisotropic and depend in a complicated way on η .⁷ Its exact dependence is of no concern here but they are isotropic $A_{\hat{x}} = A_{\hat{y}}$, and equal $A_\mu = B_\mu$, when $\eta = 1$. In the vicinity of $\eta = 1$, they can be considered as isotropic but distinct with $(A_{\text{eff}} - B_{\text{eff}}) \propto (1 - \eta)$. In this case, a Migdal-Kadanoff renormalization-group analyses indicates a double transition of an Ising followed by an XY as temperature increases,⁷ which merge when $\eta = 1$, in agreement with the Monte Carlo simulations.⁸ We will still assume this to be valid for much larger deviations from the $\eta = 1$ case.

Now, we concentrate on the Ising transition at low temperatures for $\eta \neq 1$. Since near this transition the system has XY order, the coupled XY -Ising model in Eq. (2) renormalizes to⁷

$$\frac{H}{kT} = \frac{1}{2} \sum_{r,\mu} (A_\mu + B_\mu S_r S_{r+\mu}) (\theta_r - \theta_{r+\mu})^2 - \sum_{r,\mu} L_\mu S_r S_{r+\mu}, \quad (3)$$

where now $A_{rr'}$, $B_{rr'}$, and $L_{rr'}$ are renormalized couplings. They will be complicated functions of both η and temperature.

The helicity modulus measures the stiffness of the system against an imposed phase twist.¹³ It can be calculated as the change in the free-energy density f when a long-wavelength twist k is applied to the system, using the relation

$$\gamma = \left(\frac{d^2}{dk^2} f \right)_{k=0}.$$

For the coupled XY -Ising model in Eq. (3), it results

$$\frac{\gamma_\mu}{kT} = A_\mu - B_\mu \left(\frac{\partial}{\partial L_\mu} (\beta f) \right)_{A,B}. \quad (4)$$

At the Ising transition we have

$$\frac{d}{dL} \beta f \propto (T - T_c) \log |T - T_c|,$$

and γ_μ has a logarithmic singularity. γ_x and γ_y are distinct

FIG. 2. Helicity modulus in the x and y direction for $\eta = 0.5$ as obtained by Monte Carlo simulations in a 8×8 square lattice.

FIG. 3. Temperature and size dependence of $(d/d\beta)(\beta\gamma_y)$, near the Ising transition. Solid lines are merely guides to the eyes. Inset shows the finite-size scaling of the peak heights.

and will have additional temperature dependencies through A and B , however, these renormalized values should not add any other singularity. Thus, the temperature derivative of γ will diverge logarithmically at the Ising transition. Similar arguments have been used by den Nijs¹⁵ in the context of the antiferromagnetic restricted solid-on-solid model.

To test these results, we have computed the helicity modulus by standard Monte Carlo simulations. The helicity modulus was obtained using the relation¹

$$\gamma_\mu = \frac{1}{N^2} \left[\left\langle \sum_r J_{r,r+\mu} \cos(\theta_r - \theta_{r+\mu}) \right\rangle^2 - \frac{1}{kT} \left\langle \left(\sum_r J_{r,r+\mu} \sin(\theta_r - \theta_{r+\mu}) \right)^2 \right\rangle \right]. \quad (5)$$

In Fig. 2, we show the temperature behavior of the x and y component of this quantity, for $\eta = 0.5$. The Ising transition occurs at $kT/J \approx 0.18$ as indicated by the specific heat peaks.⁸ The effect on the helicity modulus is more pronounced in the y component, in the form of a kink. To check the logarithmic divergence of its temperature derivative, we have in addition calculated $\frac{1}{2} (d/d\beta)(\beta\gamma_\mu)$ using the internal energy difference between periodic U_p and antiperiodic U_a boundary conditions in the y direction. They are related by

$$\frac{1}{2} \frac{d}{d\beta} (\beta\gamma) = \frac{(U_a - U_p)}{\pi^2}. \quad (6)$$

In Fig. 3, we show some preliminary results of a finite-size scaling analysis of the Monte Carlo data. The results are consistent with a linear dependence of the peak height on the $\ln N$, suggesting the expected logarithmic divergence of $(d/dt)\gamma$ with temperature, near the Ising transition.

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