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TWO UPWINDING SCHEMES FOR NONLINEAR PROBLEMS IN FLUID DYNAMICS

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The appropriated modeling of convection terms is a key point for reproducing complex physical phenomena in fluid dynamics problems, particularly in hyperbolic nonlinear conservation laws and related fluid flow problems. In this context, the objective of this work is to present/compare two new high-order upwind schemes for approximating convective terms, namely the Six-Degree Polynomial Upwind Scheme of C^1 Class (SDPUS-C1) and the "Esquema teste" scheme. "Esquema teste" has the same properties of the SDPUS-C1 scheme, but it is of C^2 class. These schemes are based on normalized variable diagram of Leonard [2] and the TVD principle of Harten [1, 2]. In summary, the schemes are:

– SDPUS-C1 [2]:

$$\hat{\phi}_f = \begin{cases} (-24+4\gamma)\hat{\phi}_U^6 + (68-12\gamma)\hat{\phi}_U^5 + (-64+13\gamma)\hat{\phi}_U^4 + \\ \qquad (20-6\gamma)\hat{\phi}_U^3 + \gamma\hat{\phi}_U^2 + \hat{\phi}_U, & \hat{\phi}_U \in [0,1], \\ \hat{\phi}_U, & \hat{\phi}_U \notin [0,1], \end{cases} \tag{1}$$

where γ is a free parameter; or in flux-limiter notation,

$$\psi(r) = \max\left\{0,\ \frac{0.5(|r|+r)(4r^2+12r)}{(1+|r|)^4}\right\}, \tag{2}$$

where $r = \hat{\phi}_U/[1-\hat{\phi}_U]$ is a local shock sensor.

– "Esquema teste":

$$\hat{\phi}_f = \begin{cases} -4(\lambda-24)\hat{\phi}_U^8 + 16(\lambda-23)\hat{\phi}_U^7 + (528-25\lambda)\hat{\phi}_U^6 \\ +(19\lambda-336)\hat{\phi}_U^5 + (80-7\lambda)\hat{\phi}_U^4 + \lambda\hat{\phi}_U^3 + \hat{\phi}_U, & \hat{\phi}_U \in [0,1], \\ \hat{\phi}_U, & \hat{\phi}_U \notin [0,1], \end{cases} \tag{3}$$

where λ is a free parameter; or in flux-limiter notation

$$\psi(r) = \frac{0.5(|r|+r)[(2\lambda-32)r^4 + (160-4\lambda)r^3 + 2\lambda r^2]}{(1+|r|)^7}. \tag{4}$$

The performance of the SDPUS-C1 and "Esquema teste" schemes are assessed by solving 1D/2D Burgers, Shallow Water and Euler equations. The schemes are then used for simulating incompressible fluid flow involving moving free surfaces. In particular, in this abstract, we present only some numerical results for 1D/2D Euler equations. These problems are simulated by using the CLAWPACK code of LeVeque [1] with the Godunov method equipped with a correction term containing as the flux-limiter. When the Monotinized Centered (MC) of van Leer (see [1] pp. 115) is employed as the flux-limiter, we obtain the reference solution for this work; and when our SDPUS-C1 (with $\gamma = 12$) and "Esquema teste" (with $\lambda = 96$) schemes are used as flux-limiters, we derive the numerical solutions.

– 1D Euler equations: these equations are given by $\phi_t + F(\phi)_x = 0$, where $\phi = [\rho, \rho u, E]^T$ is the vector of conserved variables and $\mathbf{F}(\phi) = [\rho u, \rho u^2 + p, u(E+p)]^T$ is the flux function vector; ρ, u, ρu, E and p are the density, the velocity, the momentum, the total energy and the pressure, respectively. The ideal gas equation of state $p = (\gamma-1)(E - \frac{1}{2}\rho u^2)$ is consider to close the system with $\gamma = 1.4$. We consider a Riemann problem involving multiple interactions of strong shocks. The problem is solved in $[0,1]$ with initial conditions

$$[\rho_0, u_0, P_0]^T = \begin{cases} [1, 0, 1000]^T, & \text{if } \ 0 \le x \le 0.1, \\ [1, 0, 0.01]^T, & \text{if } \ 0.1 < x \le 0.9, \\ [1, 0, 100]^T, & \text{if } \ 0.9 < x \le 1 \end{cases} \tag{5}$$

and zero-order extrapolation as boundary conditions. This Riemann problem is solved using a mesh size of 2000 computational cells, at Courant number $\theta = 0.5$, to obtain the reference solution, and solved using a mesh size of 1000 computational cells, at $\theta = 0.8$, for the numerical solutions. Figure 1 depicts the reference and numerical solutions for ρ, at time $t = 0.38$. We observed, from this figure, that both schemes SDPUS-C1 and "Esquema teste" are in a good agreement with the reference solution, being that the "Esquema teste" scheme provided the best numerical result.

– 2D Euler equations: these equations are given by $\phi_t + F(\phi)_x + G(\phi)_y = 0$, where the vector of conserved variable is $\phi = [\rho, \rho u, \rho v, E]^T$ and the flux functions are $F(\phi) = [\rho u, \rho u^2 + p, \rho uv, (E+p)u]^T$ and $G(\phi) = [\rho v, \rho uv, \rho v^2 + p, (E+p)v]^T$; $[u, v]^T$ and $[\rho u, \rho v]^T$ being the velocity and momentum vectors, respectively. In order to close the system, we consider $p = (\gamma-1)(E - \frac{1}{2}\rho(u^2+v^2))$, with $\gamma = 1.4$. Another interesting Riemann problem is the interation of two oblique shocks (states a and c) with two normal shocks (states b and d) (see [2]). This problem is solved in

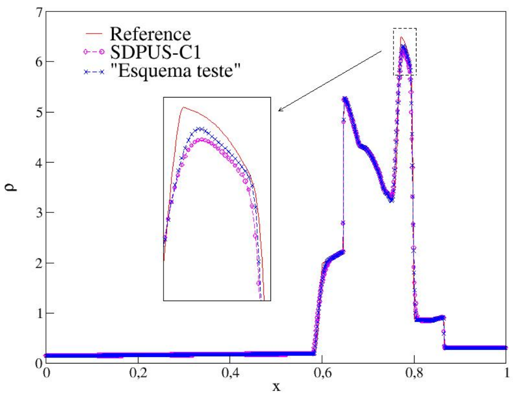

Figure 1 – 1D Euler equations: comparison of solutions for ρ.

domain $[0,1]\times[0,1]$, with initial conditions

$$[\rho_0,u_0,v_0,p_0]^T = \begin{cases} [1.5,0,0,1.5]^T & \text{a,}\\ [0.137,1.206,1.206,0.029]^T & \text{b,}\\ [0.532,1.206,0,0.3]^T & \text{c,}\\ [0.532,0,1.206,0.3]^T & \text{d} \end{cases} \quad (6)$$

and zero-order extrapolation as boundary conditions. The reference (at $\theta = 0.5$) and numerical (at $\theta = 0.8$) solutions are solved in a mesh size of 200×200 computational cells. We report in Figure 2 the solutions for p on $y = x$ line, and in Figure 3 the p contours, at $t = 0.8$. One can see, from these figures, that both the schemes solved satisfactory the complex structure, with the "Esquema teste" scheme providing the best result.

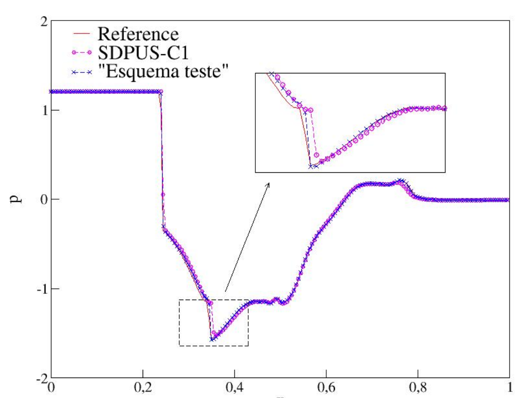

Figure 2 – 2D Euler equations: solutions for p on $y = x$.

From this short presentation, one can conclude that the SDPUS-C1 and "Esquema teste" schemes are good tools for capturing shocks and complex structures. For the future, we will compare these upwinding schemes in the scenario of incompressible fluid flow involving free surfaces. We thank the financial support given by FAPESP (Grants 2008/07367-9 and 2009/16954-8) and CNPq (Grants 300479/2008-5).

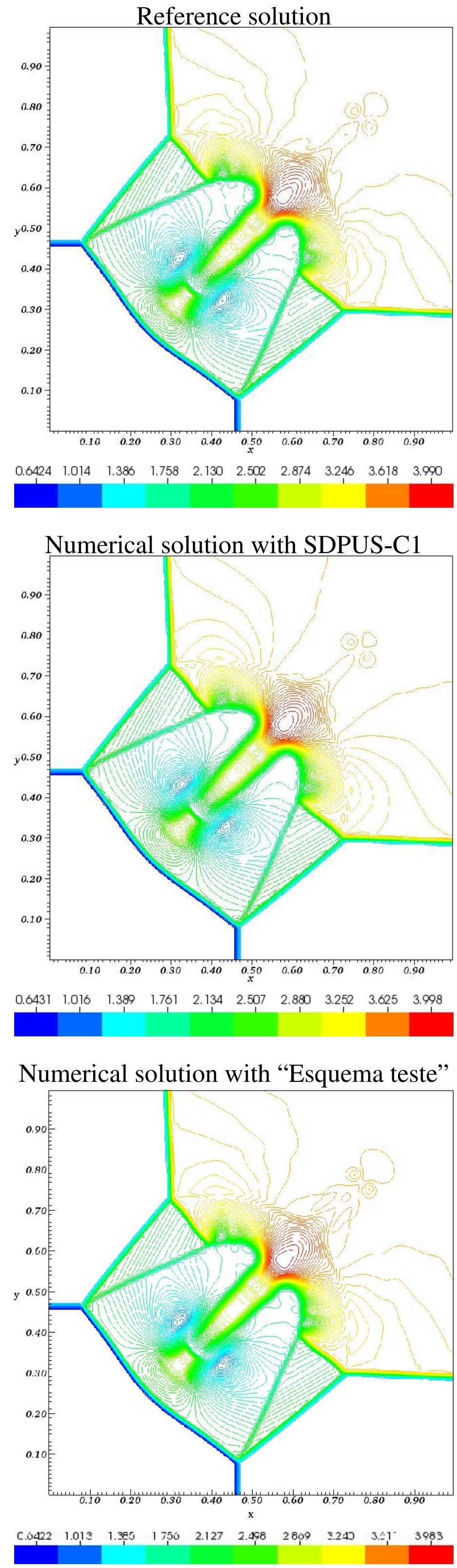

Figure 3 – 2D Euler equations: solutions of the contours of p.

References

[1] R. J. LeVeque, "Finite volume methods for hyperbolic problems," Press Syndicate of the University of Cambridge, 2002.

[2] G. A. B. Lima, "Desenvolvimento de estratégias de captura de descontinuidades para leis de conservação e problemas realcionados em dinâmica dos fluidos", Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Brasil, 2010.